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The Logic Of Fundamental Processes: Nonmeasurable Sets And Quantum Mechanics

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THE LOGIC OF FUNDAMENTAL PROCESSES: NONMEASURABLE SETS AND QUANTUM MECHANICS

by

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ABSTRACT

Quantum theory has played a significant role in modern philosophy both as a source of metaphysical ideas and as an important example of a 'scientific revolution'. In spite of the sixty or so years that have elapsed since its invention, a long lasting controversy concerning the interpretation and meaning of quantum theory prevails. Almost all authors, however, seem to agree on one major point, namely, that there could be no interpretation of this theory which is both realistic and local.

The purpose of this thesis is to demonstrate that this premiss is false and that a realistic, local and deterministic interpretation of quantum theory (at least of part of it) does exist, provided that we extend the classical concept of probability.

In order to establish this a 'quasi classical' probability theory is developed based on some non Lebesgue measurable 'events', which is then applied to account for spin-statistics. Finally I note how this model reflects on the problems of physical realism, locality, the status of probability theory and the philosophical foundations of mathematics.

ACKNOWLEDGMENTS

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I am also deeply indebted to Dr. W. Demopoulos and Dr. W. Harper for frequent discussions of various topics and for their constant encouragement, and to my colleague, M. Forster for his penetrating criticism.

I have discussed my model with a few physicists whose comments helped me in shaping my ideas. I am grateful in particular to Dr. P. Moldauer from Argonne National Laboratory (Argonne IL) who gave me my first chance to present my ideas to physicists, to Dr. N. D. Mermin from the Atomic and Solid State Laboratory at Cornell (Ithaca NY) and to Dr. H. P. Stapp from the Lawrence Berkeley Laboratory (Berkeley CA).

Last but not least I wish to thank my wife, Liora Lurie, who followed the 'ups' and 'downs' of my career with constant encouragement and significant personal sacrifice. This thesis is dedicated to her.

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CHAPTER I

INTRODUCTION

It is a matter of fact that there are no universally accepted foundations for the theory of probability, and it is generally agreed that quantum mechanics employs a rather peculiar concept of probability. This is part of the reason why Feynman "can safely say that nobody understands quantum mechanics".[1] One can, as Feynman and others do, blame Nature for this: "The behaviour of things on the very tiny scale is simply different . . . They behave in a way that is nothing that you have seen before."[2] In this thesis I shall suggest that in some profound sense 'tiny things' are just like macroscopic objects, and it is not Nature that is responsible for the confusion but rather our own concepts of probability vis à vis physical reality.

Quantum mechanics (as opposed to the old quantum theory) was introduced in 1925 by the then young and relatively unknown physicist, Werner Heisenberg. His short paper: "On Quantum Theoretical Interpretation of Kinematical and Mechanical Relations"[3] was immediately recognized as revolutionary. At that time Niels Bohr was already a prominent and very famous physicist, considered second only to Einstein in his generation (indeed he was). In the thirty or so years following the invention of quantum mechanics Bohr coordinated and guided its development. His direct contribution to the rapidly developing mathematical formalism was only secondary (compared with the work of

Heisenberg, Shrödinger, Born, Dirac, and Von Neuman); he was rather more interested in the fundamental metaphysical puzzles that the new theory brought about. For better or worse, Bohr's interpretation of quantum mechanics has dominated physics ever since. Endowed with a strong conviction and tremendous personal charisma, he managed to convert most of the young prominent physicists to his camp. In one famous incident he 'twisted Heisenberg's arm' until the latter accepted his views.[4]

The differences between Heisenberg's earlier position and Bohr's own views are striking. Heisenberg believed that a *measurement* of the position of a particle induces a disturbance in the value of its momentum (and vice versa). Thus, position and momentum cannot be 'operationally' defined simultaneously. This was indeed a crucial observation for Heisenberg who, as a positivist, preferred operational definitions. Heisenberg's views, however, did not exclude the logical possibility that a more elaborate theory could predict what *precisely* happens to the momentum of a particle as a result of a position measurement. It is at least logically possible that the knowledge of the initial values of some parameters (hidden variables) will make it possible to determine the simultaneous position-momentum values in some indirect way. It is true that such an extended theory cannot be tested directly, but it may have indirect observable consequences.

Bohr excluded this possibility on a priori grounds. Position and momentum are not simply 'operationally' undefined simultaneously; rather, they do not exist as observables independently of measurements that establish their values. Such measurements do not merely uncover properties of the particle; rather, they 'bring into existence' the very properties which are observed. This interpretation, argued Bohr, is supported if not explained by the dual 'wave-particle' nature of microscopic objects. It follows that a deterministic extension of quantum theory is unphysical, since such a theory attempts to ascribe properties to particles independently of their measurement and therefore commits an ontological fallacy.

This famous Copenhagen Interpretation is still the view which is promoted by most textbooks on quantum mechanics. I believe it to be inherently obscure and by and large unjustified. Indeed, the idea that physical properties exist only when observed is so metaphysically bizarre that many physicists take a rather cynical view of it. When asked for his opinion of the Copenhagen Interpretation, Dirac said something like: "I believe it is very important, very important indeed, especially for a student who is taking the final examination".[5]

There are, of course, some very powerful arguments against the possibility of a deterministic extension of quantum theory. These, however, came onto the scene only in the sixties. Apart from a result by Von Neumann[6], which was later shown[7] to be limited in scope, there were no convincing arguments for rejecting a deterministic view on a priori grounds. The urge to see Nature through the spectacles of uncertainty did not result from a careful logical analysis. What, then, were the historical reasons for this development?

Few historians have attempted to answer this difficult question. In a long and puzzling article Forman[8] has suggested that external factors, such as the political and cultural milieu of the Weimar Republic, played a significant role in the evolution of the Copenhagen Interpretation. It is not my intention to evaluate the historical accuracy of these controversial claims, nor am I qualified to do so.

CHAPTER III

SPHERICAL STATISTICS

1. Definitions and Notations

As before, x,y,z,w will denote unit vectors in three dimensional Euclidean space, $S^{(2)}$ the set of all unit vectors and $x \cdot y$ the scalar product of $x,y \in S^{(2)}$. Let m be the normalized Lebesgue measure on $S^{(2)}$ (so that $m(S^{(2)}) = 1$). For a fixed $z \in S^{(2)}$ and fixed angle $0 < \theta < \pi$, let $c(z,\theta)$ denote the set of all unit vectors that form an angle θ with z, that is, $c(z,\theta) = \{w \in S^{(2)} \mid w \cdot z = \cos\theta\}$. $c(z,\theta)$ is a circle on $S^{(2)}$ whose radius is $\sin\theta$ and center on the axis connecting -z with z. Let $m_{z\theta}$ be the Lebesgue measure *on the circle* $c(z,\theta)$ so that $m_{z\theta}[c(z,\theta)] = 2\pi \sin\theta$. Denote $p_{z\theta} = (2\pi\sin\theta)^{-1} m_{z\theta}$, then $p_{z\theta}$ is a normalized (probability) measure on $c(z,\theta)$.

Definition (3-1): Let f be a real function defined on $S^{(2)}$. f is *spherically integrable* if for all $z \in S^{(2)}$ and $0 < \theta < \pi$ the restriction of f to $c(z,\theta)$ is $p_{z\theta}$ integrable. In such a case we shall denote

$$E(f \mid z,\theta) = \int_{c(z,\theta)} f(w)\,dp_{z\theta}(w). \tag{3-1}$$

$E(f \mid z,\theta)$ is the *conditional expectation* of f on $c(z,\theta)$. Also, if the integral

$$E(f \mid z) = \frac{1}{2}\int_0^\pi E(f \mid z,\theta)\sin\theta\, d\theta \tag{3-2}$$

attitude of modern physicists towards fundamental issues was not necessarily irrational either. What appears as cynicism or crude instrumentalism is, sometimes, a natural reaction to the pressures of the times.

In any case, the situation today is different. The interest in fundamental problems has revived, in particular after the publication of a result by J. S. Bell. It is clear today that understanding the quantum mechanical concept of probability is the key to a solution of the problem of interpretation. The predictions of quantum mechanics are, in part, phrased in probablistic terms. This fact in itself does not entail any particular view of reality. The results of a completely deterministic chain of events (e.g., tossing a coin) could also be described in the language of probability theory. What makes quantum mechanics a theory that 'nobody understands' is not the mere fact that probablistic statements are utilized. The trouble, rather, is that the concept of 'probability' as employed by this theory deviates considerably from all other standard uses of probability theory.

This is an elementary observation which was, surprisingly, ignored or de-emphasized by the founders of quantum mechanics. Max Born, who won the Nobel Prize for his probablistic interpretation of the 'wave function', hardly mentioned the difference between quantum mechanical and standard probability when he came to summarise his philosophical views.[12] I believe that Feynman[13] was the first prominent physicist to stress this obvious difference.

The point is that in standard uses of probability outside quantum mechanics, there is a clear relation between expectation values on the one hand and relative frequencies on the other. Take as a simple example the coin toss case. Suppose that we are given a coin and after examining it with the appropriate instruments we arrive at the conclusion that the metal from which it is made is evenly distributed. Invoking a symmetry principle, we infer that the expectation of the outcome 'heads' in an arbitrary toss of the coin is 0.5. Indeed, in a long sequence of tosses we are likely to find that the outcome 'heads' occurs with frequency 0.5. This conformity of expectation values with frequencies is the content of the law of large numbers,[14] which states that the means of a random sequence of identically distributed random variables converge, most likely, to their mutual expectation. The law of large numbers does not amount to a justification of the axioms of probability, nor does it reduce the concept of 'expectation' to 'frequency'. Such uses of this law are clearly circular.[15] Its role is rather more modest, but nevertheless significant. For example, if we take the coin case as a probablistic model of the real physical event 'tossing a coin made of evenly distributed metal' then, *within such a restricted model*, the law of large numbers serves as a *theoretical explanation* of the fact that expectation values, which reflect a physical property of the coin (symmetry), conform to concrete observable results (relative frequencies).

In quantum mechanics the situation is different. The theory predicts expectation values for outcomes of experiments on the basis of physical principles (not least among these are symmetry principles similar to the one invoked in the coin case). These expectations, in turn, agree beautifully with observed frequencies. What we lack, though, is an explanation *why*. If we invoke standard probablistic models in an attempt to find such an explanation we get wrong results. For this is the central enigma of quantum phenomena: One always counts too many or too few particles than one would expect on the basis of *any* traditional probablistic argument. In such cases, physicists talk about 'interference effects' and blame the dual particle-wave nature of the elementary constituents of matter. This, however, is hardly an explanation; it is merely a repetition of Feynman's claim that 'tiny things are simply different'.

In the coin case probablistic models are dispensable (in principle) because, in the framework of classical mechanics, we can determine the outcome of any toss given the initial conditions. Probablistic models are invoked precisely in those cases when, for some practical reason, we cannot control these initial conditions. Quantum mechanics, by contrast, is inherently probablistic, that is, causes (initial conditions) do not always determine a unique effect. The question is whether we can extend this theory and make it deterministic. Maybe it is the case that we do not know all the relevant initial conditions, maybe there are yet undetected physical parameters (hidden variables) which, together with the known observables do determine a unique outcome.

Any attempt to construct such an extension faces the difficulty of the 'interference effects'. In any such framework one should be able to explain why quantum mechanics, as it is today, works. But, as I have mentioned before, quantum mechanics defies all our intuitions regarding probability. Thus, if we attempt to explain quantum statistics by averaging over a random distribution of hidden variables (initial conditions), utilizing in the process the law of large numbers, we shall not get the correct result.

This fact seems to suggest that quantum mechanics is indeed irreducibly probablistic ('probablistic', though, in an ill understood sense of the term). The purpose of this thesis is to point towards another possible explanation. I suggest that our intuitions regarding probability might be wrong-headed and that this is the source of the difficulties. In chapters III-V I shall construct an extension of probability theory based on some mathematical 'pathologies' (non-measurable sets) and show that with such abstract creatures one can consistently extend quantum theory (at least part of it) in a deterministic fashion. Moreover, in the framework of this proposed model the agreement between expectations and frequencies is explained by an appropriate version of the law of large numbers.

Chapter II is devoted to an exposition of the difficulties that quantum mechanices poses and chapter VI to the philosophical implications of the proposed model for the foundations of physics, the theory of probability and the problem of realism in mathematics.

CHAPTER II

BELL'S INEQUALITY

1. The Spin

Non-local behaviour on the part of elementary particles has been known for decades, the classical example being the two slit experiment with photons (or other elementary particles). If one assumes that photons are localized classical particles, small balls, so to speak, one cannot explain the interference that occurs in the experiment. In fact, a classical particle description of the experiment entails that in some cases a single photon passes simultaneously through the two slits,[1] and this leads to absurdities. Both the classical wave theoretical[2] and the quantum mechanical[3] explanations of the two slit experiment are very complicated, involving calculations over many degrees of freedom. As a result, it is extremely difficult to isolate those parameters which behave non-classically and arrive at the root of the problem.

Perhaps the most striking and simple example of apparently non-local effects is provided by the spin correlations between identical particles. Bell,[4] elaborating on ideas of Einstein-Podolsky-Rosen[5] and Böhm,[6] has shown that correlations exist that cannot be explained if we assume a classical picture of particles. As we shall see below, Bell's observations have more far-reaching implications. The advantage of Bell's example is mainly its simplicity. It involves only one degree of freedom (the spin), which is easily handled owing to the fact that its quantum mechanical analysis involves only finite dimensional Hilbert spaces. I shall thus concentrate on this degree of freedom, hoping that a better understanding of the so-called Einstein-Podolsky-Rosen paradox may shed light on more complicated phenomena as well.

The spin, or intrinsic angular momentum, was introduced into physics in order to explain the behaviour of free particles in magnetic fields. In a sequence of experiments conducted during 1921-1922, O. Stern and W. Gerlach[7] showed that when a beam of silver atoms is passed through a uniformly directed magnetic field it splits into two sub-beams of approximately the same intensity. From the dimensions of the split it appeared that each one of the atoms carries angular momentum of magnitude $\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$ directed parallel to the magnetic field. (Here $\hbar$ is Planck's constant divided by 2π. In the following I shall adopt a unit system in which $\hbar = 1$.) The two possible signs of this angular momentum value explain the split that occurs in the beam. Various experiments with different particles have established the fact that each particle carries spin which is either $\pm$ half integer or $\pm$ integer (zero included). Fairly quickly it became apparent that the observable spin (toghether with angular momentum) is involved in the explanation of a vast number of physical phenomena such as polarization, scattering processes, magnetism. More recently it has been shown that the spin is by and large 'responsible' for the strange behaviour of matter at low temperatures. The importance of the spin in quantum statistics stems from a superselection rule called 'Pauli's Exclusion Principle'[8] which is, in fact, a generalization of the famous rule originally introduced by Pauli in order to explain the periodic table. This rule states that the wave function associated with a system of identical particles is either symmetrical or anti-symmetrical (with respect to a permutation of the particles) according to whether the particles have integral or half integral spin, respectively.

The existence of the spin degree of freedom posed a theoretical problem for quantum mechanics since it does not emerge as a possible quantum number (degree of freedom) from the solutions of the Schrödinger equation. In the 1930s, however, Dirac constructed a synthesis of quantum mechanics with special relativity and demonstrated that in a relativistic quantum mechanics the spin is a 'natural' quantum number. Indeed, in the framework of contemporary quantum field theory, the spin of a particle by and large determines the character of the field equation associated with it.[9] All this shows that the examples of spin statistics with which I shall deal are not merely esoteric isolated cases, but rather focus directly on the heart of the matter.

The term 'spin' (as well as the term 'angular momentum') is rather unfortunate and misleading and there are two reasons for this. Firstly, it implies that the particle actually rotates, which leads immediately to difficulties. For suppose that a proton does rotate with angular momentum $\frac{1}{2}\hbar$ then, taking the proton size into account, one can easily prove that the velocity on its surface is greater than the velocity of light. Hence the rotation assumption leads to difficulties with special relativity. The second and, for my purpose, more important reason why the term 'spin' is misleading has to do with its quantized nature. When we pass a random beam of electrons through a Stern-Gerlach magnet the beam splits into two sub-beams with identical intensity, no matter in which orientation the magnet happens to be. Moreover, the size of the split is also independent of the particular orientation. Thus it appears that the electron has angular momentum $\pm\frac{1}{2}$ in all directions, so that it 'rotates' simultaneously around all possible axis. For these reasons it is advisable to take the term 'spin' with a grain of salt and consider it at best as a metaphor that bears some formal similarities with classical angular momentum.

In the analysis that follows I shall take the electron as a paradigm case of spin $-\frac{1}{2}$ particles. All my observations apply equally well to all other spin $\frac{1}{2}$ particles. Generalizations to higher spin states will be developed in chapter V.

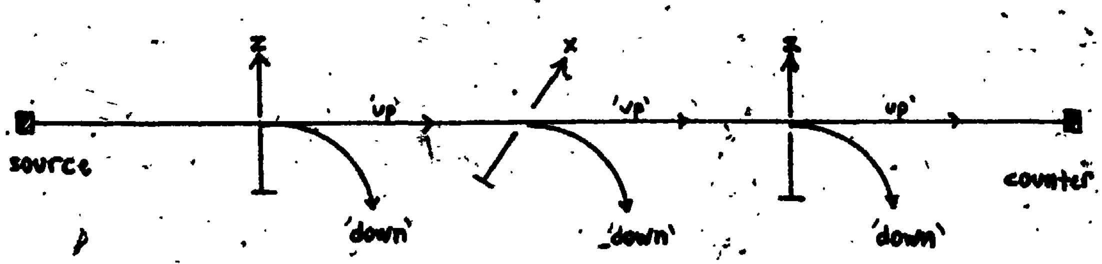
2. The Incommensurability Thesis for the Spin

Polarization of electron beams by Stern-Gerlach magnets seems to indicate that at every given moment each electron carries with it a definite spin, which is $+\frac{1}{2}$, or $-\frac{1}{2}$ in every direction. Let x,y,z,w denote directions in physical space (that is, unit vectors in three dimensional Euclidean space) and let $S^{(2)}$ be the set of all directions (that is, the surface of the unit sphere in three-dimensional Euclidean space). If we ignore quantum mechanics for the moment and consider the above natural assumption regarding the spin, we are led to the following:

Definition (2-1): A spin function is a function $s : S^{(2)} \to \{-\frac{1}{2},\frac{1}{2}\}$ which satisfies $s(-w) = -s(w)$ for all unit vectors $w \in S^{(2)}$. Suppose that we associate with every given electron at every given moment a spin function which assigns every direction either spin 'up' (that is, $+\frac{1}{2}$) or 'down' $(-\frac{1}{2})$. The identity $s(-w) = -s(w)$ follows from the fact that the spin values in opposite directions have opposite signs. This realist assumption leads immediately to difficulties, which result from the following highly confirmed statement:

Operational Incomensurability Thesis: A measurement of the electron spin in a given z direction may 'flip' the spin value in other, non-opposite, directions.

A simple experimental verification[10] of this thesis is as follows: We take a beam of electrons and pass it through a Stern-Gerlach magnet oriented in the z direction. Next we take the sub-beam of the original beam which is polarized 'up' and pass it through a second apparatus oriented in the x direction, for $x \neq \pm z$. Finally, we take again the 'up' polarized sub-beam that results and pass it through a third apparatus oriented in the z direction (Fig. 1).



(Fig. 1)

If the second polarization, in the x direction, does not disturb the spin values in the z direction then all the electrons that go through the third apparatus should have spin 'up'. This, however, is not the case. For example, if x is orthogonal to z, about 12% of the original beam will go 'down' through the third apparatus.

Thus, even if we assume that an electron has a definite spin in all directions there seems to be no obvious way to determine its value directly in more than one direction. Prima facie, this fact seems to be sufficient evidence for believing in the following.

Epistemic Incomensurability Thesis: We can never know the simultaneous spin value of an electron in two (or more) non-opposite directions.

An extreme positivist may even take a stronger position since, in his view, magnitudes that are indeterminable in princple have dubious ontological status. This, I suspect, is part of the reason for believing in the

Copenhagen Incommensurability Thesis: The 'observable 'spin in the z direction' does not exist unless measured. A spin measurement in a given direction z brings into existence the very property which is observed, namely the spin value in the z direction.

The epistemic incommensurability thesis (let alone the Copenhagen thesis) is much stronger than the operational thesis, because prima facie it is possible that the spin value of the electron in each given direction z is uniquely determined by the values of some parameters, either those that already appear in the theory (momentum, angular momentum, the spin in another direction, etc.) or else parameters that are yet to be discovered. If this set of parameters includes some that are incommensurable with the spin in the z direction (e.g., the spin in another direction), then our extended theory cannot be tested directly because of the operational incommensurability thesis. Nevertheless, it may have indirect observable consequences. Therefore, unless it has been shown that such a theory could not exist in princple, the epistemic incommensurability thesis cannot be taken as valid.

The first candidate for a parameter that determines the spin value in the $\pm z$ direction is the spin value in some other direction. It might be the case that the spin in one direction fixes, by some rule, the spin in another, or even all other directions. Such one-one correlations, if they exist, would enable us to calculate the simultaneous spin in two or more directions on the basis of one measurement only. Again, we cannot check such a theory directly (because of the operational incommensurability thesis), but it is easy to devise an indirect experiment that will test our claim. I shall show below that such a theory is impossible on a priori grounds. To be more precise, I shall prove that the existence of one-one correlations between spins in different (non-opposite) directions is incompatible with the isotropy of physical space. Let s_0 be a spin function associated with an electron. The function s_0 describes a physically possible distribution of spin values in all directions. Let O_3 be the group of orthogonal transformations in three dimensional Euclidean space. For $\alpha \in O_3$ define the spin function $s_0 \circ \alpha$ by $s_0 \circ \alpha(w) = s_0(\alpha(w))$ for all $w \in S^{(2)}$. $s_0 \circ \alpha$ is clearly a spin function. Since space is isotropic we must conclude that if s_0 is a physically possible spin function so is $s_0 \circ \alpha$, for all $\alpha \in O_3$. Denote $F_0 = \{s_0 \circ \alpha \mid \alpha \in O_3\}$. The assumption that the spin value in a given direction z alone fixes the spin value in another direction x means that either $s(z) = s(x)$ for all physically possible spin functions s, or else $s(z) = -s(x)$ for all physically possible spin functions. Since $-s(x) = s(-x)$ this means that there are two directions z, y (where $y = x$ or $y = -x$) such that $s(z) = s(y)$ for all physically possible spin functions s. Contrary to this we have:

<u>Theorem (2-1)</u>: *The set* F_0 *separates the points of* $S^{(2)}$, *that is, for any given pair of directions* y, z, $y \neq z$ *there is a function* $s \in F_0$ *such that* $s(y) \neq s(z)$.

<u>Proof</u>: Suppose, by negation, that there is a pair y, z such that $y \neq z$ and $s(y) = s(z)$ for all $s \in F_0$. Since $s_0(-z) = -s_0(z)$ it follows that $y \neq -z$, hence y, z are linearly independent. Let P be the plane spanned by y and z, and let β be a rotation of P such that $\beta(y)$ lies between y and z, that is, $\measuredangle(y,\beta(y)) + \measuredangle(\beta(y),z) < \pi$. Let γ be an arbitrary rotation around the y axis, that is, $\gamma(y) = y$, and let $s \in F_0$. We have $s(y) = s(\gamma(y)) = s \circ \gamma(y)$. Since $s \circ \gamma \in F_0$ we get $s(y) = s \circ \gamma(y) = s \circ \gamma(z)$. Let δ be any rotation around the $\beta(y)$ axes. We have $\delta(\beta(y)) = \beta(y)$, hence for all $s \in F_0$: $s(\beta(y)) = s(\delta\beta(y)) = s \circ \delta\beta(y)$. Since $s \circ \delta\beta \in F_0$ we get $s(\beta(y)) = s \circ \delta\beta(y) = s \circ \delta\beta(z) = s \circ \delta(\beta(z))$. The vector $\beta(y)$ lies between y and z, hence there are rotations γ (around the y axes) and δ (around the $\beta(y)$ axes) such that $\gamma(z) = \delta(\beta(z)) = w$ (Fig. 2).

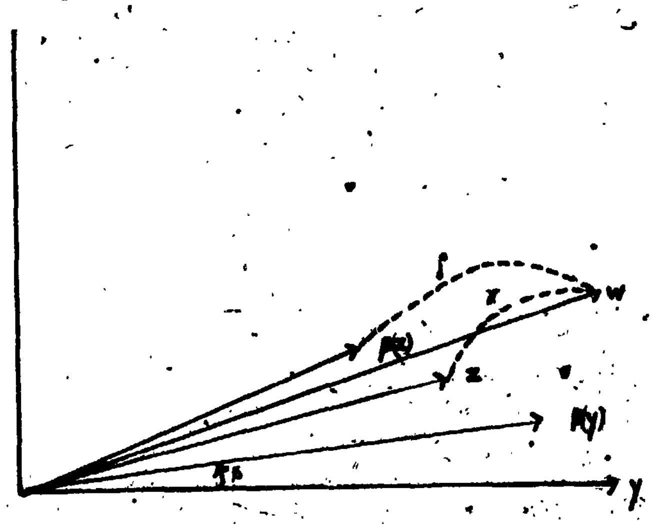


Fig. 2

Therefore we have for all $s \in F_0$:

$$s(y) = s \circ \gamma(z) = s(w) = s \circ \delta(\beta(z)) = s(\beta(y)) = s \circ \beta(y).$$

Hence, $s \circ \beta(y) = s(y)$, therefore $s \circ \beta^2(y) = s \circ \beta(\beta(y)) = s \circ \beta(y) = s(y)$ (since $s \circ \beta \in F_0$). In the same way we can prove $s \circ \beta^n(y) = s(y)$ for all $n = 1,2,\ldots$. Now choose β such that for some integer n, β rotates the plane P by $\frac{\pi}{n}$. If n is sufficiently large, then $\beta(y)$ lies between y and z and thus we get $s \circ \beta^n(y) = s(y)$. But $\beta^n(y) = -y$, hence $s(-y) = s(y)$, a contradiction. Q.E.D.

We see from the above simple geometrical theorem that no hidden variable theory could exist which determines the spin values of an electron only on the basis of other (non-opposite) spin values. This fact does not rule out, of course, the possibility that a more elaborate hidden variable theory that includes more parameters will do the job.

3. The Einstein-Podolsky-Rosen Experiment

The gap between the operational incommensurability thesis and the epistemic incommensurability thesis was the focus of Einstein, Podolsky, and Rosen's (E.P.R.) classic paper.[5] It is true, they argued, that experimental constraints prevent us from *measuring directly* the simultaneous values of certain pairs of observables (such as position-momentum, or a pair of spin values in two non-opposite directions, etc.). In some cases, however, those which involve composite systems of two (or more) particles, one can gain knowledge of simultaneous values of such incommensurable pairs. For example, when a particle spontaneously decays into two other particles, A and B, one can gain knowledge of the simultaneous values of both the position and the momentum of A by directly measuring its position, and by inferring its momentum from a measurement performed on B (utilizing the law of conservation of momentum). Thus it seems that the epistemic incommensurability thesis is false and the Copenhagen interpretation ungrounded.

The experiments proposed in the E.P.R. paper remained thought experiments until Böhm[6] devised a particularly simple version of the argument which involves spin values (rather than momentum) and which gives rise to experiments that can actually be carried out. In certain cases, two interacting electrons emerge from the interaction in the so-called 'singlet state'. This means operationally that the two electrons have opposite spins in every direction in which spin is measured. To verify that this is the case one takes a source that emits electron pairs in the singlet state. The electrons that emerge from the source form two beams that travel in opposite directions, left and right. Each electron on the left beam has a singlet state companion on the right beam and vice versa. Next, one polarizes the two beams by Stern-Gerlach magnets located at the same distance on both sides of the source and oriented in the same direction z (Fig. 3).

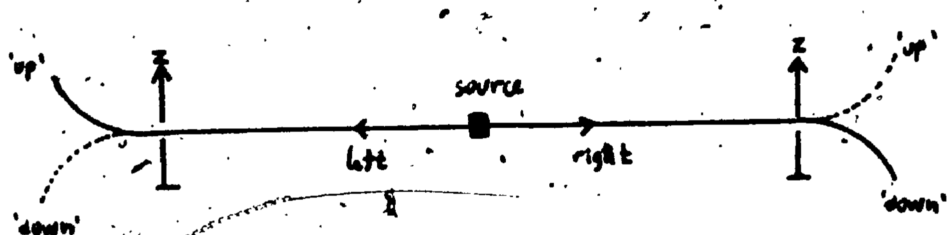


Fig. 3

If we count the simultaneous registrations of 'spin up on the left and spin down on the right', and simultaneous registrations of 'spin down on the left and spin up on the right', we observe that the left side electron has spin up if and only if its singlet state companion on the right has spin down. This phenomenon occurs no matter which direction z we choose. If we return to our realist picture and maintain that each electron at each given moment has definite spin in all directions, we conclude:

Left-Right Inference Rule: A pair of electrons in the singlet state has opposite spins in all directions.

Is this rule true? Even if we accept the premise that electrons do have definite spins in all directions, this rule does not follow from the results of the E.P.R. Böhm experiment alone. It might be the case that the two electrons do not have opposite spins in all directions, but as a result of the measurement performed on the left side a fast signal is sent to the right side that causes the right electron to stay in, or flip to, the appropriate spin value. (On the basis of symmetry we can assume that a similar signal is sent from right to left.) Thus, the opposite spins are observed because an interference has occurred in the experiment, and not because the spins were opposite *before* the measurement.

It seems reasonable to rule out such an interference on the basis of physical principles. The distance between the left and right magnets can be made as large as we wish (at least in principle).[11] Any known random interference should vanish (for all practical purposes) at large distances and, more importantly, any known signal takes time to propagate across such distances. Therefore, we can make the distance between the magnets sufficiently large so that any familiar signal from left to right will arrive at its target only after polarization on the right has already occurred. We may conclude, therefore, and maintain:

The Principle of Locality: The right hand measurement does not influence the results of the left hand measurement, and vice versa.

Only if we accept the principle of locality can we maintain that the left-right inference rule has indeed been established by the E.P.R. Böhm experiment. It seems, however, reasonable to maintain this.

Take, next, an experimental set-up which is similar to the one we have considered before, only this time the magnets are oriented in different directions: z on the left and x on the right (Fig. 4).

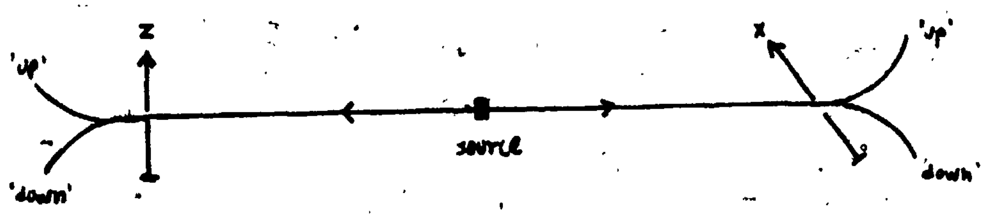


Fig. 4

In this case, we may count simultaneous registrations of 'spin up in the z direction on the left and spin down in the x direction on the right'. Now we use the left-right inference rule and conclude that the spin in the x direction on the right is down iff it is up on the left. Hence we have devised an experiment that establishes the simultaneous spin values in both the z and x directions for the left electron. Thus, the conjunction of the properties of the singlet state and the principle of locality establishes a counterexample to the epistemic incommensurability thesis. Such counterexamples, argued Einstein, Podolsky, and Rosen, indicate that quantum mechanics does not give us a complete description of physical reality. Quantum mechanics deals only with one definite spin value at a given moment, while the rest are assumed to be non-definite (each definite spin state in one direction is a non-trivial superposition of the two spin states in another direction). On the other hand, the spin values in two directions seem to be definite, at least in some cases which are not accounted for by the formalism of quantum mechanics. This calls for an extension of the theory that might turn out to be deterministic.

4. Bell's Inequality

"The paradox of Einstein-Podolsky- and Rosen was advanced as the argument that quantum mechanics could not be a complete theory but should be supplemented by additional variables. These additional variables were to restore to the theory causality and locality. In this note that idea will be formulated mathematically and shown to be incompatible with the statistical predictions of quantum mechanics".

This is the opening statement of Bell's paper.[4] Indeed, the very possibility of an E.P.R. experiment seems to support the realist assumption that the electron spin is definite in two (or even all) directions, independently of whether a measurement is performed or not. It seems also to support the speculation that a more complete and deterministic description of microscopic events might exist. The irony is that the *statistical* results of the E.P.R. Bohm experiment strongly indicate that such a realist assumption regarding the spin is inconsistent and that a hidden variable theory is impossible. Thus the observations made in the E.P.R. and Böhm papers turn out to be an argument against the point of view they proposed.

For any two directions x,z let p(z,x) be the limit of the relative frequency of 'spin up in the z direction on the left and spin down in the x direction on the right', as measured in an appropriate E.P.R. experiment. If we accept the left-right inference rule, then p(z,x) is identical with the frequency of 'spin up in the z direction and spin up in the x direction' for the left electron. In particular, for x = z the number p(z,z) is the frequency of 'spin up in the z direction'.

From the properties of relative frequencies it follows that the quantities p(z,x) should satisfy the following conditions:

a) $p = p(z,x)$ is a function from the set of pairs of directions $S^{(2)} \times S^{(2)}$ into the interval [0,1];

b) For all $x,z \in S^{(2)}$

$$[p(-z,x) + p(z,x)] + [p(-z,-x) + p(z,-x)] = 1; \qquad (2\text{-}1)$$

c) For all z:

$$p(-z,z) = 0 \qquad (2\text{-}2)$$

since no particle has both spin up and down in a given direction.

In addition to these properties we might add the following condition, which reflects the isotropy of space:

d) The value of p(z,x) depends only on the angle between x and z. In other words:

$$p(\alpha(z),\alpha(x)) = p(z,x), \quad \text{for all } \alpha \in O_3. \qquad (2\text{-}3)$$

If condition d) were false we would be able to find a preferred pair of directions in space (those for which p(z,x) is maximal among the pairs which form the same angle). Thus we have

Definition (2-2): A function p that satisfies conditions a-d above is called a *spin correlation function* (s.c.f.). We have immediately:

Lemma (2-2): *Any* s.c.f. *satisfies* $p(z,x) = p(x,z)$ *and* $p(z,z) = \frac{1}{2}$ *for all* $z,x \in S^{(2)}$.

Proof: Since the value of $p(z,x)$ depends only on the angle between z and x, we have $p(z,x) = p(x,z)$. In particular, for $x = -z$ we get $p(z,-z) = p(-z,z) = 0$ (condition c). Thus, from condition b) for $x = -z$: $1 = p(-z,z) + 2p(z,z) + p(z,-z) = 2p(z,z)$. Hence $p(z,z) = \frac{1}{2}$ Q.E.D.

Consider now the hypothetical situation (a possible world) where the relative frequencies observed in an E.P.R. type of arrangement are given by a fixed s.c.f. $p = p(z,x)$. What does it mean to have a local hidden variable theory that explains these correlations? The most general answer seems to be the following. There exists a set Λ of physical parameters (some of them may be unknown to present-day physics). Each electron, at every given moment, is associated with a unique value $\lambda \in \Lambda$, and we say that at this moment the electron is in the 'state' λ. Given a value of λ one is able to calculate (among other things) the precise spin value of the electron in each direction at this moment. Thus with every $\lambda \in \Lambda$ there corresponds a spin function $s_\lambda = s_\lambda(x)$. Suppose that we are given a random sample of n electrons. The i^{th} electron is in the state λ_i and thus associated with the spin function s_{λ_i}. Put, for simplicity, $s_i = s_{\lambda_i}$ and let z,x be two directions. Denote by $K_n(z,x)$ the following set of indices: $K_n(z,x) = \{i \mid 1 \le i \le n, s_i(z) = s_i(x) = +\frac{1}{2}\}$, and let $\overline{K}_n(z,x)$ be the cardinality of $K_n(z,x)$.

We have

Definition (2-3): The s.c.f. p admits a local hidden variable representation on a set Λ if for every 'random' sequence $\lambda_1, \lambda_2, \ldots \lambda_n$ the corresponding sequence of spin functions $s_1, s_2, \ldots, s_n$ 'most probably' satisfies

$$n^{-1} \bar{K}_n(z,x) \xrightarrow[n\to\infty]{} p(z,x) \qquad (2\text{-}4)$$

for all z and x.

This definition could not be taken as mathematically rigorous. The key words 'random' and 'most probably' are probablistic concepts and I have not assumed that any particular probablistic structure exists on Λ. What makes Bell's argument particularly strong is the fact that it does not depend (so it appears prima facie) on any particular interpretation of these concepts. Note that the assumption of locality enters indirectly into the definition (2-3) since we identify the values of limiting relative frequencies of a composite system, as given by the s.c.f. p(z,x), and the limit of averages taken only on the left hand sequence of electrons ($\lim_{n\to\infty} n^{-1} \bar{K}_n(z,x)$). In other words, we actually use the left-right inference rule. The following results from very elementary arithmetical considerations:

Lemma (2-3): *Let* $s_1, s_2, \ldots, s_n$ *be as above and denote* $\chi_i(x) = s_i(x) + \frac{1}{2}$ *for* $1 \le i \le n$. *We have, for all* $x, y, z \in S^{(2)}$:

a) $\chi_i(x) + \chi_i(-x) = 1$

b) $\sum_{i=1}^{n} \chi_i(z)\chi_i(x) = \bar{K}_n(z,x)$

c) $\chi_i(-x)\chi_i(y) + \chi_i(-y)\chi_i(z) \ge \chi_i(-x)\chi_i(z)$.

Proof: a) $\chi_i(x) + \chi_i(-x) = s_i(x) + \frac{1}{2} + s_i(-x) + \frac{1}{2} = 1 + s_i(x) - s_i(x) = 1.$

b) $\chi_i(z)\chi_i(x) = 1$ in the case $s_i(z) = \hat{s}_i(z) = +\frac{1}{2}$, otherwise

$\chi_i(z)\chi_i(x) = 0$. Hence, by definition, $\sum_{i=1}^{n} \chi_i(z)\chi_i(x) = \bar{K}_n(z,x)$.

c) Since $\chi_i(z) + \chi_i(-z) = 1$ we have for all $x,y \in S^{(2)}$

$$\chi_i(-x)\chi_i(y) = \chi_i(-x)\chi_i(y)\chi_i(z) + \chi_i(-x)\chi_i(y)\chi_i(-z) \geq$$

$$\geq \chi_i(-x)\chi_i(y)\chi_i(z). \tag{2-5}$$

Since $\chi_i(x) + \chi_i(-x) = 1$:

$$\chi_i(-y)\chi_i(z) = \chi_i(x)\chi_i(-y)\chi_i(z) + \chi_i(-x)\chi_i(-y)\chi_i(z) \geq$$

$$\geq \chi_i(-x)\chi_i(-y)\chi_i(z). \tag{2-6}$$

Adding inequalities (2-5), (2-6) and utilizing the fact that $\chi_i(y) + \chi_i(-y) = 1$, we get:

$$\chi_i(-x)\chi_i(y) + \chi_i(-y)\chi_i(z) \geq \chi_i(-x)\chi_i(z). \tag{2-7}$$

Q.E.D.

Thus we have:

Theorem (2-4) (Bell): *A necessary condition that a* s.c.f. $p = p(z,x)$ *admits a local hidden variable representation on any set* Λ *is that the inequality*

$$p(-x,y) + p(-y,z) \geq p(-x,z) \tag{2-8}$$

will be satisfied for all x,y,z.

Proof: Averaging over inequality (2-7) and utilizing Lemma (2-3 b) we get

$$n^{-1}\,\overline{K}_n(-x,y) + n^{-1}\overline{K}_n(-y,z) =$$

$$= n^{-1}\sum_{i=1}^{n} \chi_i(-x)\chi_i(y) + n^{-1}\sum_{i=1}^{n} \chi_i(-y)\chi_i(z) \geq$$

$$\geq n^{-1}\sum_{i=1}^{n} \chi_i(-x)\chi_i(z) = n^{-1}\overline{K}_n(-x,z) .$$

Taking the limit $n \to \infty$ we get

$$p(-x,y) + p(-y,z) \geq p(-x,z) . \qquad \text{Q.E.D.}$$

Bell's inequality in the above form contains three pairs of directions (xy, yz, xz) and any actual physical measurement is always performed on one pair of directions at a time. It is easier to understand the meaning of Bell's argument if we derive from it another inequality that refers only to one pair of directions. To do this put $p(z,x) = p(\theta)$ whenever θ is the angle between z and x. Let $\theta_1,\theta_2,\theta_3$ be the angles between xy, yz, and xz, respectively. The angle between -x and y is $\pi - \theta_1$. Hence,

$p(\pi - \theta_1) + p(\theta_1) = p(-x,z) + p(x,z) = \frac{1}{2}\,[p(-x,z) + 2p(x,z) + p(x,-z)] = \frac{1}{2}$,

or $p(\pi - \theta_1) = \frac{1}{2} - p(\theta_1)$. The same applies to θ_2,θ_3. Thus Bell's inequality could be written as

$$p(\pi - \theta_1) + p(\pi - \theta_2) \geq p(\pi - \theta_3)$$

or

$$p(\theta_1) + p(\theta_2) - p(\theta_3) \leq \frac{1}{2} \qquad (2\text{-}9)$$

whenever $\theta_1,\theta_2,\theta_3$ satisfy $\theta_3 \leq \theta_1 + \theta_2$. Having established this we can prove:

Theorem (2-5): *A necessary condition that a* s.c.f. $p(\theta) = p(z,x)$ *admits a local hidden variable representation on* *<u>any</u>* *set* Λ *is that*

$$p(\tfrac{\pi}{n}) \le \tfrac{1}{2}(1 - \tfrac{1}{n}) \qquad (2\text{-}10)$$

for $n = 1,2,3\ldots$.

Proof: We have $np(\frac{\pi}{n}) = p(\frac{\pi}{n}) + p(\frac{\pi}{n}) + (n-2)p(\frac{\pi}{n})$. Put $\theta_1 = \theta_2 = \frac{\pi}{n}$ and $\theta_3 = \frac{2\pi}{n}$. We get from (2-9)

$$np(\tfrac{\pi}{n}) \le \tfrac{1}{2} + p(\tfrac{2\pi}{n}) + (n-2)p(\tfrac{\pi}{n}) = \tfrac{1}{2} + p(\tfrac{2\pi}{n}) + p(\tfrac{\pi}{n}) + (n-3)p(\tfrac{\pi}{n}).$$

Iterating the process for $\theta_1 = \frac{2\pi}{n}$, $\theta_2 = \frac{\pi}{n}$, and $\theta_3 = \frac{3\pi}{n}$ we get

$$np(\tfrac{\pi}{n}) \le 2 \times \tfrac{1}{2} + p(\tfrac{3\pi}{n}) + (n-3)p(\tfrac{\pi}{n}) .$$

After $n-1$ steps we have

$$np(\tfrac{\pi}{n}) \le \frac{n-1}{2} + p(\tfrac{n\pi}{n}).$$

Hence, since $p(\frac{n\pi}{n}) = p(\pi) = p(-x,x) = 0$, we have

$$p(\tfrac{\pi}{n}) \le \tfrac{1}{2}(1 - \tfrac{1}{n}). \qquad \text{Q.E.D.}$$

Until this stage I have considered a hypothetical spin correlation function p and derived some necessary conditions for recovering the frequencies given by p in a local hidden variable theory. Quantum mechanics predicts, and experiments verify, that the actual s.c.f. in physical reality is $p_0(\theta) = \frac{1}{2}\cos^2(\frac{\theta}{2})$ (or $p_0(z,x) = \frac{1}{4} + \frac{x \cdot z}{4}$, where $x \cdot z$ is the scalar product of x and z). Let $\theta = \frac{2\pi}{3}$, then $p_0(\theta) = \frac{1}{8}$.

On the other hand let p be *any* s.c.f. *which does have a local hidden variable representation.* We get $p(\frac{2\pi}{3}) = p(\pi - \frac{\pi}{3}) \geq \frac{1}{2} - p(\frac{\pi}{3})$, and thus from (2-10): $p(\frac{2\pi}{3}) \geq \frac{1}{6}$.

Consider an E.P.R. experiment in which the angle between the magnets is $\frac{2\pi}{3}$. We can expect on the basis of *any* local hidden variable theory that in a random sample of 10^6 electron pairs more than 165,000 will have the property 'spin up on the right hand side and spin down on the left hand side'. What we observe *in fact* is that the number is approximately 125,000 -- a substantial difference.

5. What Has Gone Wrong?

The existing answers to this question range from technical analyses of experiments to mystical Zen type interpretations. I shall not attempt to survey the vast literature on this subject. In the concluding chapter I shall deal with some philosophical aspects of various answers. At this stage I shall briefly describe some of the more common attitudes.

i) Nothing has gone wrong: There are various positions which allow one to maintain this view. For the Copenhagen approach the violation of Bell's inequality is yet another proof that "the moon is not there when nobody looks".[12] If one assumes a classical analysis of physical properties one cannot recover spin correlations but if, on the other hand, the very existence of the electron spin depends on somebody looking at it no contradiction arises. This position maintains some flavour of realism. Its adherents do not deny that electrons actually exist (together with their properties). What makes this view non-classical are the conditions under which such ontological claims are sound.

For the anti-realist, that is, a person who takes an instrumentalist view with regard to theories of elementary particles, the E.P.R. paradox is a treasure: it is a celebrated demonstration that no reasonable realist position exists. If the only way to save realism involves metaphysical gymnastics (as in the Copenhagen Interpretation) or obscure physical theories (maintaining a violation of locality, see below), then it seems that this is the only plausible position left.[13]

ii) The Locality Principle is Wrong: And thus also the left-right inference rule. There have been some suggestions that mysterious 'superluminal' signals might carry the message from one space location to another instantaneously, and consequently that our present day conceptions of space-time might be naive.[14] There seems to be no reason to reject such a theory on a priori grounds, and indeed it is possible that the principle of locality is a mere approximation. At this stage there are no detailed suggestions except for a few ad hoc arguments that do the job in the E.P.R. case.

It is an interesting sociological fact that those who support a non-local approach in the realm of physics usually preach a holistic mystical metaphysics. If every object in the universe influences every other object instanteously and the influence does not diminish with distance, then it is not clear what 'object' means. The universe, so the story goes, is the only object that has a sound onthological status, while parts of the universe (particles, stones, chairs, galaxies) are mere linguistic approximations. There is no identity criterion for parts of the universe since what appears to us as a chair, say, is the product of a complex and irreducible interaction of rules that govern the universe as a whole.

Some people[15] take the violation of Bell's inequality as a triumph of the holistic Zen view over what they call the Western philosophical tradition (by which they mean both the Cartesian and Empiricist views). Surprisingly (or perhaps not so) this mystical position is very similar to the romantic Hegelian metaphysics which is certainly part of the Western tradition and which was described by Bertrand Russell[16] in the following words: "The [Hegelian material] world is like a jelly in the fact that, if you touched any part of it, the whole quivered, but unlike a jelly in that it could not be cut into pieces".

I believe that this position deserves the name: 'The California Interpretation of Quantum Mechanics'. Its adherents have done more than any other group to popularize the subject.

iii) Classical Logic is False: The state of a system is described by reference to its properties as defined in an appropriate theory. The propositions that depict this state are called 'state descriptions'. If we assume that the logic that governs state descriptions is different from the classical propositional calculus, in particular that it violates the axiom of distributivity, then one can formally 'explain' why classical analysis fails to capture interference phenomena. This approach was introduced by Birhoff and Von Neumann[17] and is referred to as 'quantum-logic'. In this logic the family of physical events (or state descriptions) is not a Boolean algebra and thus formal probability functions defined on such an 'event space' violate the axiom of additivity without which Bell's inequality cannot be proved.

Putnam[18] has promoted the view that logic, like the geometry of space time, is empirical and that quantum mechanics 'forces' upon us a non-classical logic in the same way that general relativity 'forces' upon us a non-Euclidean geometry. Syntactically, the quantum logician can recover the appropriate probabilities (which is not a surprise given that the 'event space' in quantum logic is the lattice of closed subspaces of a Hilbert space). The trouble lies, however with the semantics of quantum logic. It is not at all clear why quantum logic is different. By contrast general relativity does explain why (and how much) the geometry of space-time deviates from the Euclidean picture. Without such an explanation quantum logic can hardly be taken as an 'explanation'. The view which I shall promote below is intimately related to quantum logic, but is different in that it does not take the logic as given, rather explaining, on the basis of probablistic arguments, the reasons why it is different.

iv) Probability Theory is False: Or rather should be extended.[19] I have already mentioned that Feynman was the first to stress the difference between the quantum mechanical use of the term 'probability' and all it's classical uses. He argues that quantum theory gives us a set of rules to calculate probabilities and thus to make predictions. 'Nobody understands' why these rules work, but for Feynman, a rather extreme instrumentalist (who despises philosophers), this is not a real obstacle. What I shall propose in the following is a realistic and local reduction of the quantum mechanical rules to those of a probability theory which is 'almost classical'. The only non-classical component in my approach is the use of 'pathological' non-Lebesgue measurable distributions.[20]

CHAPTER III

SPHERICAL STATISTICS

1. Definitions and Notations

As before, x,y,z,w will denote unit vectors in three dimensional Euclidean space, $S^{(2)}$ the set of all unit vectors and $x \cdot y$ the scalar product of $x,y \in S^{(2)}$. Let m be the normalized Lebesgue measure on $S^{(2)}$ (so that $m(S^{(2)}) = 1$). For a fixed $z \in S^{(2)}$ and fixed angle $0 < \theta < \pi$, let $c(z,\theta)$ denote the set of all unit vectors that form an angle θ with z, that is, $c(z,\theta) = \{w \in S^{(2)} \mid w \cdot z = \cos\theta\}$. $c(z,\theta)$ is a circle on $S^{(2)}$ whose radius is $\sin\theta$ and center on the axis connecting $-z$ with z. Let $m_{z\theta}$ be the Lebesgue measure *on the circle* $c(z,\theta)$ so that $m_{z\theta}[c(z,\theta)] = 2\pi \sin\theta$. Denote $p_{z\theta} = (2\pi \sin\theta)^{-1} m_{z\theta}$, then $p_{z\theta}$ is a normalized (probability) measure on $c(z,\theta)$.

Definition (3-1): Let f be a real function defined on $S^{(2)}$. f is *spherically integrable* if for all $z \in S^{(2)}$ and $0 < \theta < \pi$ the restriction of f to $c(z,\theta)$ is $p_{z\theta}$ integrable. In such a case we shall denote

$$E(f \mid z,\theta) = \int_{c(z,\theta)} f(w)\,dp_{z\theta}(w). \qquad (3\text{-}1)$$

$E(f \mid z,\theta)$ is the *conditional expectation* of f on $c(z,\theta)$. Also, if the integral

$$E(f \mid z) = \frac{1}{2}\int_0^{\pi} E(f \mid z,\theta) \sin\theta \, d\theta \qquad (3\text{-}2)$$

is defined for some z it is called the z-*directional expectation* of f. If $E(f \mid z)$ is defined for all z and its value is independent of z then f is said to be *totally spherically integrable* and $E(f) = E(f \mid z)$ is the *total spherical expectation* of f. Let $A \subseteq S^{(2)}$ be a set. Define $\chi_A(w) = 1$ if $w \in A$ and 0 if $w \notin A$. A is *spherically measurable* if χ_A is spherically integrable, and A is *totally spherically measurable* if χ_A is totally spherically integrable. I shall denote $E(A \mid z,\theta) = E(\chi_A \mid z,\theta)$, $E(A \mid z) = E(\chi_A \mid z)$, and $E(A) = E(\chi_A)$ whenever these quantities are well defined.

Every bounded Borel function on $S^{(2)}$ is totally spherically integrable and its total spherical expectation is just its integral with respect to the normalized Lebesgue measure m on the sphere. To see this, let z be arbitrary and let (r,θ,ϕ) be a spherical coordinate system for which $z = (1,0,0)$. Then

$$\int_{S^{(2)}} f(w)dm(w) = \frac{1}{4\pi}\int_0^{\pi}\left[\int_0^{2\pi} f(\theta,\phi)d\phi\right]\sin\theta\, d\theta =$$

$$= \frac{1}{2}\int_0^{\pi} E(f \mid z,\theta)\sin\theta\, d\theta = E(f \mid z).$$

But since f is a Borel function its integral is independent of a particular choice of coordinates. Thus $E(f \mid z) = E(f)$ is the total spherical expectation of f.

Not every bounded totally spherically integrable function is a Borel or even Lebesgue function. We shall see that the family of all totally spherically measurable sets is not even a Boolean algebra. Thus there are two totally spherically measurable sets A, B such that $A \cap B$ is not totally spherically measurable.

Let $A \subseteq S^{(2)}$ be an arbitrary set. The (normalized) *outer measure* of A is defined by:

$$\overline{m}(A) = \inf \{m(G) \mid G \supset A,\ G \text{ open}\}$$

where 'open' refers to the standard Euclidean topology on $S^{(2)}$. For every Lebesgue set A we have $m(A) = \overline{m}(A)$.[1] For spherically measurable sets we have

<u>Lemma (3-1)</u>: *Let* A *be spherically measurable, then*

$$\overline{m}(A) \geq \sup_{z} E(A \mid z) \tag{3-3}$$

where the supremum is taken over all directions z *for which* $E(A \mid z)$ *is defined.*

<u>Proof</u>: Let z be a direction for which $E(A \mid z)$ is well defined and let G be an open set such that $G \supset A$. Then $G \cap c(z,\theta) \supset A \cap c(z,\theta)$ and $G \cap c(z,\theta)$ is open relative to $c(z,\theta)$ for all $0 < \theta < \pi$. Hence

$$m(G) = \frac{1}{2}\int_0^\pi E(G \mid z,\theta)\sin\theta\, d\theta \geq \frac{1}{2}\int_0^\pi E(A \mid z,\theta)\sin\theta\, d\theta = E(A \mid z).$$

This is true for all the open sets G such that $G \supset A$, hence $\overline{m}(A) \geq E(A \mid z)$. Since the last inequality holds for all z for which $E(A \mid z)$ is defined we have proved the claim. Q.E.D.

2. <u>Fermi Functions</u>

In the previous chapter I have introduced the concept of a spin function. In this section I shall confine myself to a special type of spin function:

<u>Definition (3-2)</u>: A Fermi function is a function $s : S^{(2)} \to \left\{-\frac{1}{2},\frac{1}{2}\right\}$ which satisfies $s(-w) = -s(w)$ for all $w \in S^{(2)}$, s is spherically integrable, and for all $z \in S^{(2)}$ and $0 < \theta < \pi$:

$$E(s \mid z,\theta) = s(z) \cos \theta. \tag{3-4}$$

I have to prove first that this definition is not void, that is, that Fermi functions exist. This claim turns out to be true if we accept as valid some 'heavy' set theoretical axioms. By this I mean firstly the axiom of choice, or rather its equivalent the principle of well ordering, and secondly the following axiom which relates measure to cardinality and which I shall call the measure-theoretic continuum hypothesis: (M.C.H.):

M.C.H. *Every subset of a Lebesgue space whose cardinality is strictly less than* $2^{\aleph_0}$ *is Lebesgue measurable and has Lebesgue measure zero.*

Where by a 'Lebesgue space' I mean a measure space isomorphic with the interval [0,1] and the Lebesgue measure on it. The M.C.H. is clearly consistent relative to Zermelo-Fraenkel set theory since it follows trivially from the continuum hypothesis. It is however, much weaker than the continuum hypothesis, since it follows also from Martin's axiom.[2] Having these set theoretic assumptions we can prove:

<u>Theorem (3-2)</u>: *There exists a Fermi function.*

<u>Proof</u>: For a fixed $z \in S^{(2)}$ and $0 < \theta \le \frac{\pi}{2}$ denote $R(z,\theta) = c(z,\theta) \cup c(-z,\theta) \cup \{-z,z\}$. The family $K = \{R(z,\theta) \mid z \in S^{(2)}$, $0 < \theta \le \frac{\pi}{2}\}$ has the power of the continuum and thus, by the principle of well-ordering, there is a well-ordering of K which is isomorphic to Ω,

the first ordinal whose cardinality is $2^{\aleph_0}$. $K = \{R(z_\lambda,\theta_\lambda) \mid \lambda < \Omega\}$. Let $R_\lambda = R(z_\lambda,\theta_\lambda)$ and let p_λ be the normalized Lebesgue measure on $c_\lambda = c(z_\lambda,\theta_\lambda)$. Define the function s by induction on the order:

1) First step: If $0 < \theta_1 < \frac{\pi}{2}$ divide the circle c_1 into two arbitrary disjoint subsets whose measures are $\cos^2\left(\frac{\theta_1}{2}\right)$ and $\sin^2\left(\frac{\theta_1}{2}\right)$, respectively, and put $s(z_1) = +\frac{1}{2}$ and $s(w) = +\frac{1}{2}$ for all w in the larger set (the set with the measure $\cos^2\left(\frac{\theta_1}{2}\right)$). For $w \in c(-z_1,\theta_1) \cup \{-z_1\}$ put $s(-w) = -s(w)$. If $\theta_1 = \frac{\pi}{2}$ we have $R(z_1,\theta_1) = c(z_1, \frac{\pi}{2}) = c_1$. Divide c_1 into two disjoint subsets, each with measure $\frac{1}{2}$, and such that w is in one set iff -w is in the other. Put $s(z_1) = \frac{1}{2}$, $s(w) = +\frac{1}{2}$ for w on one of the subsets, and $s(w) = -s(-w)$ for $w = -z$, and w in the other set.

2) Assume we have defined s on $\bigcup_{\sigma<\lambda} R_\sigma$ for some ordinal $\lambda < \Omega$. If

$R_\lambda = R_\sigma$ for some $\sigma < \lambda$ s has already been defined on $\bigcup_{\sigma\le\lambda} R_\sigma$.

Otherwise s has already been defined on the following subset of points of R_λ:

$$R_\lambda \cap \bigcup_{\sigma<\lambda} R_\sigma = \bigcup_{\sigma<\lambda} (R_\lambda \cap R_\sigma).$$

The intersection of two non-identical circles contains at most two points. Also the cardinality of λ is strictly less than $2^{\aleph_0}$. It follows that the cardinality of $\bigcup_{\sigma<\lambda} (R_\lambda \cap R_\sigma)$ is strictly less than $2^{\aleph_0}$, and thus also the cardinality of $\bigcup_{\sigma<\lambda} (c_\lambda \cap R_\sigma)$. Hence from M.C.H. we get that $\bigcup_{\sigma<\lambda} (c_\lambda \cap R_\sigma)$ is p_λ measurable and has measure *zero*. Therefore we can divide $c_\lambda \setminus \bigcup_{\sigma<\lambda} (c_\lambda \cap R_\sigma)$ into two disjoint subsets whose measures are $\cos^2\left(\frac{\theta_\lambda}{2}\right)$ and $\sin^2\left(\frac{\theta_\lambda}{2}\right)$ and define s in the following way: If $z_\lambda \in \bigcup_{\sigma<\lambda} R_\sigma$ and $s(z_\lambda) = -\frac{1}{2}$ define $s(w) = +\frac{1}{2}$ on the *small* subset of c_λ, $s(w) = -\frac{1}{2}$ on the larger subset, and for $w \in -c_\lambda \cup \{z_\lambda\}$ put $s(w) = -s(-w)$. In all other cases define s in the same way as in stage 1.

Completing this transfinite induction for all $\lambda < \Omega$ we define a function s on $S^{(2)}$ such that

$$E(s \mid z,\theta) = s(z)\left(\cos^2(\tfrac{\theta}{2}) - \sin^2(\tfrac{\theta}{2})\right) = s(z)\cos\theta. \qquad \text{Q.E.D.}$$

Every Fermi function is totally spherically integrable, since for all $z \in S^{(2)}$:

$$E(s) = E(s \mid z) = \frac{1}{2}\int_0^\pi E(s \mid z,\theta)\sin\theta\, d\theta =$$

$$= \frac{s(z)}{2}\int_0^\pi \cos\theta \sin\theta\, d\theta = 0\ . \tag{3-5}$$

Let $A_+ = \{w \in S^{(2)} \mid s(w) = \frac{1}{2}\}$, then we have for all $w \in S^{(2)}$: $\chi_{A_+}(w) = s(w) + \frac{1}{2}$. Thus

$$E(A_+ \mid z,\theta) = E(s+\tfrac{1}{2} \mid z,\theta) = s(z)\cos\theta + \frac{1}{2} = \begin{cases} \cos^2(\frac{\theta}{2}) & \text{if } s(z) = \frac{1}{2} \\ \sin^2(\frac{\theta}{2}) & \text{if } s(z) = -\frac{1}{2} \end{cases} \tag{3-6}$$

Therefore the set A_+ is totally spherically integrable and

$$E(A_+) = E(A_+ \mid z) = \frac{1}{2}\int_0^\pi E(s+\tfrac{1}{2} \mid z,\theta)\sin\theta\, d\theta = \frac{1}{2}. \tag{3-7}$$

A_+ is indeed totally spherically measurable, but it is not Lebesgue measurable. In fact we have:

Theorem (3-3): $\overline{m}(A_+) \geq \frac{3}{4}$.

Proof: Let z be such that $s(z) = +\frac{1}{2}$ and let G_z be the open hemisphere about z, that is, $G_z = \{w \in S^{(2)} \mid 0 < w \cdot z \leq 1\}$. Then

$$E(A_+ \cap G_z \mid z) = \frac{1}{2}\int_0^{\pi} E(A_+ \cap G_z \mid z,\theta)\sin\theta\, d\theta = \frac{1}{2}\int_0^{\pi/2} E(A_+ \mid z,\theta)\sin\theta\, d\theta =$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos^2\left(\frac{\theta}{2}\right)\sin\theta\, d\theta = \frac{3}{8}.$$

Let $\delta > 0$ be a small positive number. Then from (3-6) we get $E(A_+ \mid -z,\delta) = \sin^2\left(\frac{\delta}{2}\right) > 0$. Thus for all $\delta > 0$ there is a direction y such that $|y + z| < \delta$ and $s(y) = +\frac{1}{2}$. Put $G_y = \{w \in S^{(2)} \mid 0 < w \cdot y \leq 1\}$. Then again $E(A_+ \cap G_y \mid y) = \frac{3}{8}$. Now put $U = (G_y \cup G_z) \setminus G_y \cap G_z$, then

$$\overline{m}(A_+) \geq \overline{m}(A_+ \cap U) \geq E(A_+ \cap G_z \mid z) + E(A_+ \cap G_y \mid y) - 2m(G_y \cap G_z) =$$

$$= \frac{3}{4} - 2m(G_y \cap G_z)$$

since for all $\delta > 0$ there is a $y \in S^{(2)}$ for which $|y + z| < \delta$ and $s(y) = +\frac{1}{2}$. The number $m(G_y \cap G_z)$ can be made arbitrary small and thus $\overline{m}(A) \geq \frac{3}{4}$. Q.E.D.

Let s be a Fermi function and let O_3 be the group of orthogonal transformations in three dimensional Euclidean space. For $\alpha \in O_3$ let $s \circ \alpha$ be defined by $s \circ \alpha(w) = s(\alpha(w))$. We have

Lemma (3-4): *a) If* s *is a Fermi function so is* $s \circ \alpha$ *for all* $\alpha \in O_3$. *That is, for all* $z \in S^{(2)}$ *and* $0 < \theta < \pi$:

$$E(s \circ \alpha \mid z,\theta) = s(\alpha(z)) \cos \theta. \tag{3-8}$$

b) If s_0 *is a fixed Fermi function the family:* $F_0 = \{s_0 \circ \alpha \mid \alpha \in O_3\}$ *is infinite.*

Proof: a) We have $E(s \circ \alpha \mid z,\theta) = \int_{c(z,\theta)} s(\alpha(w))dp_{z\theta}(w)$. When w ranges over the circle $c(z,\theta)$, $\alpha(w)$ ranges over the circle $c(\alpha(z),\theta)$. Also $s(y) = \frac{1}{2}$ for all $y \in A_+ \cap c(\alpha(z),\theta)$, a set whose measure on $c(\alpha(z),\theta)$ is $\cos^2(\frac{\theta}{2})$ or $\sin^2(\frac{\theta}{2})$ according to whether $s(\alpha(z)) = +\frac{1}{2}$ or $-\frac{1}{2}$, respectively. Also $s(y) = -\frac{1}{2}$ on the rest of $c(\alpha(z),\theta)$. Hence

$$E(s \circ \alpha \mid z,\theta) = s(\alpha(z)) \left(\cos^2(\tfrac{\theta}{2}) - \sin^2(\tfrac{\theta}{2})\right) = s(\alpha(z)) \cos \theta.$$

b) Let O_3^+ be the group of 'real' rotations, that is, 3×3 orthogonal matrices with determinant $+1$. Denote $F_0^+ = \{s_0 \circ \alpha \mid \alpha \in O_3^+\}$ and consider the representation of O_3^+ as a permutation group of the set F_0^+ as follows: $\beta \to \begin{bmatrix} s_0 \circ \alpha \\ s_0 \circ \alpha\beta \end{bmatrix}$. The kernel of this representation is a normal (invariant) subgroup of O_3^+ and it cannot be O_3^+ itself (since $s_0(-w) \neq s_0(w)$). The group O_3^+ is simple, i.e., it has no normal subgroups except itself and the trivial group. It follows that F_0 is infinite, since otherwise O_3^+ is isomorphic to a finite permutation group, which is absurd. Q.E.D.

3. Random Sequences

Suppose that we consider a fixed infinite family of Fermi functions. What is then the meaning of a 'random sample' of this family? The answer cannot be given in terms of the usual Lebesgue measure on $S^{(2)}$, since Fermi functions are non-measurable. It turns out that the relevant concepts of 'independence', and consequently of 'randomness', are captured by the following:

Definition (3-3): a) Two Fermi functions are said to be *spherically independent* if for all $z \in S^{(2)}$ and $0 < \theta < \pi$:

$$E(s_1 s_2 \mid z,\theta) = E(s_1 \mid z,\theta)E(s_2 \mid z,\theta) = s_1(z)s_2(z)\cos^2\theta. \tag{3-8}$$

b) A sequence $s_1, s_2, s_3, \ldots$ of Fermi functions is said to be 'random' if for all $z \in S^{(2)}$, except maybe finitely many, and for all $0 < \theta < \pi$:

$$E(s_{i_1} s_{i_2} \ldots s_{i_k} \mid z,\theta) = \prod_{j=1}^{k} E(s_{i_j} \mid z,\theta) = (\cos\theta)^k \prod_{j=1}^{k} s_{i_j}(z) \tag{3-9}$$

for all $k \geq 2$ and choices of indices $0 < i_1 < i_2 < \ldots < i_k < \infty$. I shall prove first that these definitions are not void, that is, that random sequences exist. In fact I shall prove more:

Theorem (3-5): *There is an orthogonal transformation* α *and a Fermi function* s *such that* $\{s \circ \alpha^n\}_{n=1}^{\infty}$ *is a random sequence.*

Proof: Let y_0 be a fixed direction and let α be an orthogonal transformation such that $\alpha(y_0) = y_0$, and such that α rotates the plane orthogonal to y_0 by an angle ϕ for which $\phi\pi^{-1}$ is an irrational number.

As in the proof of theorem (3-2) put $R(z,\theta) = c(z,\theta) \cup c(-z,\theta) \cup \{-z,z\}$ for $z \in S^{(2)}$ and $0 < \theta \leq \frac{\pi}{2}$. Denote $L(z,\theta) = \bigcup_{n=-\infty}^{\infty} R(\alpha^n(z),\theta)$.

We have:

(i) $L(y_0,\theta) = R(y_0,\theta)$ for all $0 < \theta \leq \frac{\pi}{2}$.

(ii) If $z' = \pm\alpha^k z$ for some $k = 0,\pm 1,\pm 2\ldots$ then $L(z,\theta) = L(z',\theta)$ for all $0 < \theta \leq \frac{\pi}{2}$.

(iii) If $z' \neq \pm\alpha^k z$ for all $k = 0,\pm 1,\pm 2\ldots$ then $L(z,\theta) \cap L(z',\theta)$ is a denumerable set.

Let $z \in S^{(2)}$, $z \neq \pm y$ and $0 < \theta \leq \frac{\pi}{2}$ be fixed and consider the set $L(z,\theta)$, ignoring for a while the rest of $S^{(2)}$. I shall prove that there is a spin function s *defined on* $L(z,\theta)$ such that

$$E(s \circ \alpha^{i_1}\, s \circ \alpha^{i_2} \ldots s \circ \alpha^{i_k} \mid z,\theta) = (\cos\theta)^k \prod_{j=1}^{k} s(\alpha^{i_j}(z))$$

for all choices of indices $-\infty < i_1 < i_2 \ldots < i_k < \infty$.

To see this, well-order the countable family $\{R(\alpha^n z,\theta) \mid n = 0\ \pm 1\ \pm 2\ \ldots\}$. Suppose that the order is $R(\alpha^{n_1}(z),\theta)\ R(\alpha^{n_2}(z),\theta) \ldots R(\alpha^{n_k}(z),\theta)\ \ldots$. Put $R_j = R(\alpha^{n_j}(z),\theta)$ and define s by the induction on the order.

On R_1 define s as in the first step in the proof of theorem (3-2). Suppose we have defined s on $\bigcup_{j=1}^{k-1} R_j$. s is not defined on R_k except for the points of the subset $R_k \cap \bigcup_{j=1}^{k-1} R_j$, which is finite. Thus again we can divide the circle $c(\alpha^{n_k}z,\theta)$ into two disjoint subsets of measure $\cos^2(\frac{\theta}{2})$, $\sin^2(\frac{\theta}{2})$ and proceed as in theorem (3-2). The only extra component is that we have to make sure that our choice is such that

$$E(s \circ \alpha^{n_{i_1}} s \circ \alpha^{n_{i_2}} \dots s \circ \alpha^{n_{i_j}} s \circ \alpha^{n_k} \mid z,\theta) =$$

$$= E(s \circ \alpha^{n_{i_1}} \mid z,\theta) E(s \circ \alpha^{n_{i_2}} \mid z \dots E(s \circ \alpha^{n_{i_j}} \mid z,\theta) E(s \circ \alpha^{n_k} \mid z,\theta)$$

for all indices $i_1 < i_2 < i_3 \dots < i_j < k$. This is clearly possible, since up to the k^{th} stage s is defined only on a finite subset of $C(\alpha^{n_k} z,\theta)$, and thus the choice is arbitrary from a measure-theoretic standpoint.[3]

To complete the proof, well-order the family $\{L(z,\theta) \mid z \in S^{(2)}, 0 < \theta < \pi\}$ and define s by induction on the order. At each stage use the above construction. Since $L(z,\theta) \cap L(z',\theta)$ is denumerable for $z' \neq \pm\alpha^k(z)$, $k = 0 \pm 1 \pm 2 \dots$, and $L(z,\theta) = L(z'\theta)$ if $z' = \pm\alpha^k(z)$ for some $k = 0 \pm 1 \pm 2 \dots$, at each stage of the construction s is already defined on a small subset of points of the set $L(z,\theta)$ of the next stage, and the construction is thus possible using M.C.H. The only exceptions to this procedure are the cases where $z = y_0$, in which $L(y_0,\theta) = R(y_0,\theta)$ for $0 < \theta \leq \frac{\pi}{2}$. In these cases define s as in a typical step of the proof of theorem (3-1). Q.E.D.

Before proceeding let me demonstrate some of the 'bizarre' features of Fermi functions. Let s_1, s_2 be two spherically independent spin functions. Each one of them is totally spherically integrable but their product $s_1 \cdot s_2$ *is not*, since from eq. (3-8)

$$E(s_1 s_2 \mid z) = \frac{1}{2}\int_0^\pi E(s_1, s_2 \mid z,\theta) \sin\theta \, d\theta = \frac{1}{2} s_1(z) s_2(z) \int_0^\pi \cos^2\theta \sin\theta \, d\theta =$$

$$= \frac{1}{3} s_1(z) s_2(z).$$

Hence $E(s_1 s_2 \mid z) = \frac{1}{12}$ when $s_1(z) = s_2(z)$ and $-\frac{1}{12}$ when $s_1(z) \neq s_2(z)$. Denote $A_i^+ = \{w \mid s_i(w) = +\frac{1}{2}\}$ for $i = 1,2$. Then A_1^+ and A_2^+ are both totally spherically measurable. It is however, easy to see that $A_1^+ \cap A_2^+$ is not totally spherically measurable.

4. Laws of Large Numbers and Bell's Theorem

In standard probability theory the relation between *finite* means of identically distributed random variables and their mutual expectation value is given by a law of large numbers.[4] In the present situation a similar law applies:

Theorem (3-6): *a) Let* $s_1, s_2, \ldots, s_n \ldots$ *be a random sequence and suppose that* $s_i(z_0) = +\frac{1}{2}$ *for* $i = 1,2,\ldots$. *Then*

$$E(\{w \in S^{(2)} \mid \frac{1}{n} \sum_{i=1}^{n} s_i(w) \xrightarrow[n\to\infty]{} \frac{w \cdot z_0}{2}\} \mid z_0) = 1. \tag{3-10}$$

b) Let $s_1, s_2, \ldots, s_n$ *be a random sequence. Then for all* $z \in S^{(2)}$ *(except maybe finitely many):*

$$E(\{w \in S^{(2)} \mid \frac{1}{n} \sum_{j=1}^{n} s_j(w) \xrightarrow[n\to\infty]{} 0\} \mid z, \frac{\pi}{2}) = 1. \tag{3-11}$$

c) If $z \in S^{(2)}$ *is such that* $\frac{1}{n} \sum_{j=1}^{n} s_j(z) \xrightarrow[n\to\infty]{} 0$ *then:*

$$E(\{w \in S^{(2)} \mid \frac{1}{n} \sum_{j=1}^{n} s_j(w) \xrightarrow[n\to\infty]{} 0\} \mid z) = 1. \tag{3-12}$$

Proof: a) Let $0 < \theta < \pi$ and consider the circle $c(z_0, \theta)$. Since $s_i(z_0) = +\frac{1}{2}$ for $i = 1,2,3\ldots$ we have $E(s_i \mid z_0, \theta) = \frac{\cos\theta}{2}$ for all

$i = 1,2,\ldots,$ and since the s_i's are bivalued this also means that the s_i's are equally distributed on $c(z_0,\theta)$. From condition (3-9) it follows that the sequence $s_1, s_2, \ldots, s_n \ldots$ is independent on $c(z_0,\theta)$. Thus from the law of large numbers we conclude that $\frac{1}{n}\sum_{i=1}^{n} s_i(w) \to \frac{\cos\theta}{2}$ for almost all w on $c(z_0,\theta)$, where 'almost all' refers to the measure $p_{z_0\theta}$. Hence $E(\{\frac{1}{n}\sum_{i=1}^{n} s_i(w) \xrightarrow[n\to\infty]{} \frac{z_0 \cdot w}{2}\} \mid z_0,\theta) = 1$. Since this occurs for all $0 < \theta < \pi$ we have proved that eq. (3-10) holds.

b) For a given $z \in S^{(2)}$ divide the sequence $\{s_i\}_{i=1}^{\infty}$ into two subsets $\{s_i\} = \{s_i'\} \cup \{s_i''\}$, where $s_i'(z) = +\frac{1}{2}$ and $s_i''(z) = -\frac{1}{2}$ for $i = 1,2,\ldots$. From part a) we know that $\frac{1}{k}\sum_{i=1}^{k} s_i'(w) \to \frac{z \cdot w}{2}$ almost everywhere on $c(z,\theta)$ for all $0 < \theta < \pi$. In a similar way we can show that $\frac{1}{\ell}\sum_{i=1}^{\ell} s_i''(w) \to -\frac{z \cdot w}{2}$ almost everywhere on $c(z,\theta)$ for all $0 < \theta < \pi$. Now by definition of s_i' and s_i'' we get:

$$\frac{1}{n}\sum_{i=1}^{n} s_i(w) = \left(\frac{k}{n}\right)\left(\frac{1}{k}\sum_{i=1}^{k} s_i'(w) - \frac{z \cdot w}{2}\right) + \left(\frac{n-k}{n}\right)\left(\frac{1}{n-k}\sum_{i=1}^{n-k} s_i''(w) + \frac{w \cdot z}{2}\right)$$

$$+ \left(\frac{2k-n}{n}\right)\frac{w \cdot z}{2} \qquad (3\text{-}13)$$

When $\theta = \frac{\pi}{2}$ we have $w \cdot z = 0$ and thus the right-hand side of (3-13) converges to zero for almost all $w \in c(z, \frac{\pi}{2})$ as $n \to \infty$.

c) If $\frac{1}{n}\sum_{i=1}^{n} s_i(z) \xrightarrow[n\to\infty]{} 0$ then as n increases we get $\frac{k}{n} \sim \frac{1}{2}$ and $\frac{n-k}{n} \sim \frac{1}{2}$ in formula (3-13). We get in the limit $n \to \infty$ that

$\frac{1}{n}\sum_{i=1}^{n} s_i(w) \to 0$ almost everywhere on $c(z,\theta)$ for all $0 < \theta < \pi$, and thus

$$E(\{\frac{1}{n}\sum_{i=1}^{n} s_i(w) \xrightarrow[n\to\infty]{} 0\} \mid z) = 1. \qquad \text{Q.E.D.}$$

Note that if we apply theorem (3-6) to the random sequence that was constructed in theorem (3-5) the conclusion still holds, even for $z = y_0$. This fact follows from the ergodic theorem (not the law of large numbers), since the transformation α rotates the plane orthogonal to y_0 by an angle ϕ which is irrational modulo π.

Let $s_1, s_2, \ldots s_n \ldots$ be a random sequence and suppose $s_i(z_0) = +\frac{1}{2}$ for all $i = 1,2,\ldots$. We have proved that the set

$B = \{w \in S^{(2)} \mid \frac{1}{n}\sum_{i=1}^{n} s_i(w) \to \frac{w \cdot z_0}{2}\}$ has the following property:

$E(B \mid z_0) = 1$. Thus if we measure the size of B in polar coordinates in which z_0 is an axis it appears as if B is almost the whole sphere. Indeed, if B were measurable we would have $m(B) = \overline{m}(B) \geq E(B \mid z_0) = 1$ (Lemma (3-1)), and thus we would conclude that $\frac{1}{n}\sum_{i=1}^{n} s_i(w) \to \frac{w \cdot z}{2}$ almost everywhere. This conclusion, however, contradicts Bell's inequality. In the framework of the present model this difficulty is circumvented by the following observation.

<u>Theorem (3-7)</u>: *Let* $s_1, s_2, \ldots s_n \ldots$ *be a random sequence that satisfies, for some* $z_0 \in S^{(2)}$; $s_i(z_0) = \frac{1}{2}$ *for* $i = 1,2,\ldots$. *Then the set*

$B = \{w \in S^{(2)} \mid \frac{1}{n}\sum_{i=1}^{n} s_i(w) \xrightarrow[n\to\infty]{} \frac{w \cdot z_0}{2}\}$ *is non-measurable. Moreover, we have:* $E(B \mid z_0) = 1$, *but for all* $y_0 \in B$, $y_0 \neq \pm z_0$: $E(B \mid y_0) = 0$.

Proof: We have proved $E(B \mid z_0) = 1$. Let $y_0 \in B$, $y \neq \pm z_0$. Then put $\{s_i\} = \{s_i'\} \cup \{s_i''\}$, where $s_i'(y_0) = +\frac{1}{2}$ and $s_i''(y_0) = -\frac{1}{2}$. Denote

$$C = \left\{w \mid \frac{1}{k} \sum_{i=1}^{k} s_i'(w) \xrightarrow[k\to\infty]{} \frac{y_0 \cdot w}{2}\right\} \text{ and } D = \left\{w \mid \frac{1}{\ell} \sum_{i=1}^{\ell} s_i''(w) \xrightarrow[\ell\to\infty]{} -\frac{y_0 \cdot w}{2}\right\}.$$

Then by theorem (3-6) we have $E(C \mid y_0) = E(D \mid y_0) = 1$. Hence also $E(C \cap D \mid y_0) = 1$. On the other hand, by definition

$$\frac{1}{n} \sum_{i=1}^{n} s_i(w) = \left(\frac{k}{n}\right) \left(\frac{1}{k} \sum_{i=1}^{k} s_i'(w)\right) + \left(\frac{n-k}{k}\right) \left(\frac{1}{n-k} \sum_{i=1}^{k-n} s_i''(w)\right). \quad (3\text{-}14)$$

But, since $y_0 \in B$, we have $\frac{k}{n} = \frac{1}{n} \sum_{i=1}^{n} [\frac{1}{2} + s_i(y_0)] \to \frac{1}{2} + \frac{y_0 \cdot z_0}{2}$ and

$\frac{n-k}{n} = \frac{1}{n} \sum_{i=1}^{n} [\frac{1}{2} - s_i(y_0)] \to \frac{1}{2} - \frac{y_0 \cdot z_0}{2}$. Hence, by going to the limit $n \to \infty$ in (3-14), we conclude that $w \in B \cap C \cap D$ only if:

$$\frac{w \cdot z_0}{2} = \left(\frac{1}{2} + \frac{y_0 \cdot z_0}{2}\right) \left(\frac{y_0 \cdot w}{2}\right) + \left(\frac{1}{2} - \frac{y_0 \cdot z_0}{2}\right) \left(-\frac{y_0 \cdot w}{2}\right)$$

or $(y_0 \cdot z_0)(y_0 \cdot w) = (w \cdot z_0)$. Thus $B \cap C \cap D$ is Lebesgue measurable and has measure zero. Moreover, for all $0 < \theta < \pi$, we have $E(B \cap C \cap D \mid y_0, \theta) = 0$. Since $E(C \cap D \mid y_0, \theta) = 1$ for all $0 < \theta < \pi$ we conclude that $E(B \mid y_0, \theta) = 0$ for all $0 < \theta < \pi$ and thus $E(B \mid y_0) = 0$.

Q.E.D.

Recall that in the derivation of Bell's inequality for a sequence $s_1, s_2, \ldots, s_n \ldots$ we put $\chi_i(w) = s_i(w) + \frac{1}{2}$ for $i = 1,2,\ldots,$ and thus $\overline{K}_n(x,y) = \sum_{i=1}^{n} \chi_i(x)\chi_i(y)$ is the number of indices $1 \leq i \leq n$ for which $s_i(x) = s_i(y) = \frac{1}{2}$. Bell's inequality is

$$n^{-1}\overline{K}_n(-x,w) + n^{-1}\overline{K}_n(-w,y) \geq n^{-1}\overline{K}_n(-x,y).$$

This inequality is satisfied by every sequence of spin functions and in particular by random sequences of Fermi functions. In spite of this we have (theorem (3-6)):

$$E(\{w \mid n^{-1}\overline{K}_n(-x,w) \xrightarrow[n\to\infty]{} \tfrac{1}{2} - \tfrac{1}{2}\, x \cdot w\} \mid x) = 1$$

and

$$E(\{w \mid n^{-1}\overline{K}_n(-w,y) \xrightarrow[n\to\infty]{} \tfrac{1}{2} - \tfrac{1}{2}\, y \cdot w\} \mid y) = 1.$$

A contradiction does not occur since the set

$\{w \mid n^{-1}\overline{K}_n(-x,w) \to \frac{1}{2} - \frac{1}{2}\, x \cdot w\} \cap \{w \mid n^{-1}\overline{K}_n(-w,y) \to \frac{1}{2} - \frac{1}{2}\, y \cdot w\}$ has measure zero (theorem (3-7)).

5. Outer Measure and Probability

All I have established so far is the following fact: Given a random sequence $s_1, s_2 \ldots s_n \ldots$ of Fermi functions which satisfy $s_i(z) = \frac{1}{2}$ for all $i = 1,2,\ldots$ and some fixed $z \in S^{(2)}$, then the set $B = \{w \in S^{(2)} \mid \frac{1}{n}\sum_{i=1}^{n} s_i(w) \to \frac{w \cdot z}{2}\}$ has the property $E(B \mid z) = 1$, which also means that $\overline{m}(B) = 1$. Can we maintain that the 'event' B occurs with probability 1, that is, that the sequence of means $\frac{1}{n}\sum_{i=1}^{n} s_i(w)$ converges on $\frac{w \cdot z}{2}$ with probability 1? The answer seems at first glance negative, especially when we consider the following:

<u>Theorem (3-8)</u>: *There exists a decomposition of* $S^{(2)}$ *into* $2^{\aleph_0}$ *disjoint subsets, each having outer measure* 1.

Proof: Let H be the family of all circles $H = \{c(z,\theta) \mid z \in S^{(2)}$ $0 < \theta < \pi\}$. There is a well-ordering of H isomorphic to Ω. $H = \{c(z_\lambda,\theta_\lambda) \mid \lambda < \Omega\}$. Put $\overset{\circ}{c}_\lambda = c(z_\lambda,\theta_\lambda)$ and let p_λ be the normalized Lebesgue measure on c_λ. Define for all $\lambda < \Omega$ a set $b_\lambda \subset c_\lambda$ by induction as follows:

1) $b_1 = c_1$.

2) If $c_\lambda = c_\sigma$ for some $\sigma < \lambda$ let σ_0 be the minimal ordinal with this property and put $b_\lambda = b_{\sigma_0}$; otherwise $b_\lambda = c_\lambda \setminus \bigcup_{\sigma<\lambda} (c_\sigma \cap c_\lambda)$.

By M.C.H. we have $p_\lambda(b_\lambda) = 1$ for all λ. Now put for $z \in S^{(2)}$:

$$D_z = \cup\{b_\lambda \mid z_\lambda = z \text{ or } z_\lambda = -z\}.$$

Then $D_z \cap D_y = \emptyset$ for $y \neq \pm z$ and $D_z = D_{-z}$. Also $E(D_z \mid z) = 1$, hence $\overline{m}(D_z) = 1$ for all z. Q.E.D.

If we maintain that each set D_z in the above theorem is an 'event' that has probability one, we conclude that there are $2^{\aleph_0}$ such 'events' and the conjunction of each pair of such 'events' has probability zero, which indeed seems bizarre.

To see why it is not, consider the following situation. Suppose that we decompose the sphere $S^{(2)}$ into two disjoint *measurable* sets R,Y. The points of the set R are called red points and those of Y yellow points. Let $w \in S^{(2)}$ and assume that we want to determine whether w is red or yellow. We cannot do it by just 'looking', since no physical apparatus can determine the 'colour' of a mathematical point. What we can do, though, is to observe the intensity of red (or yellow) light coming from a small neighborhood of w. Let G_w be such a small neighborhood and suppose we have an apparatus that can filter the yellow colour while measuring the intensity of red coming from the neighborhood G_w. Suppose that this intensity is proportional to the measure of $G_w \cap R$. Then by refining our measurements, that is, taking each time a smaller neighborhood of w, we can establish the colour of w with probability 1. This follows from the fact that

$$\lim_{m(G_w)\to 0} \frac{m(G_w \cap R)}{m(G_w)} = \chi_R(w) \qquad (3\text{-}15)$$

for almost all $w \in S^{(2)}$. Therefore, in any actual physical situation, we can deduce the colour of a point within a margin of error proportional to the accuracy of our equipment.

Suppose now that the sets R,Y are non-measurable and that each of them has outer measure 1. Suppose, moreover, that the intensity of red light coming from a small neighborhood G_w of w is proportional to the *outer* measure of $G_w \cap R$. (This last assumption is reasonable since sets of outer measure 1 'fill up' every open neighborhood.) Since $\overline{m}(G_w \cap R) = m(G_w)$ we get $\dfrac{\overline{m}(G_w \cap R)}{m(G_w)} = 1$ for all w, which means that *upon a measurement* of 'red' every point of the sphere appears red and, by symmetry, upon a measurement of 'yellow' every point appears yellow. Thus the two statements 'w is red', 'w is yellow' are 'operationally' true, while their conjunction is logically false.

By analogy, I suggest understanding the formula

$$\overline{m}(\{w \mid \frac{1}{n}\sum_{i=1}^{n} s_i(w) \to \frac{w \cdot z}{2}\}) = 1$$

to mean that *upon a measurement* of the frequency $n^{-1} \sum_{i=1}^{n} s_i(w)$, in a sequence of Fermi functions $s_1, s_2, \ldots s_n \ldots$ which satisfies $s_i(z) = +\frac{1}{2}$, we are most likely to find its value close to $\frac{w \cdot z}{2}$ when n is large. Even without specifying the physical mechanism that causes this to occur, it is clear that this interpretation is consistent. Note that I have not assumed that the measurement process brings about or creates the properties which are to be measured, but only that due to some physical fact such as the inaccuracy of our equipment, measurements give the results they do.

In the following chapter I shall demonstrate that the concept of Fermi function, together with the definitions of spherical independence and randomness (Def. (3-3)), amount to an interpretation of the quantum mechanical formalism for the electron spin, and that the above interpretation of theorem (3-6) accounts for various statistical phenomena that occur in large samples of electrons. Moreover I shall derive some predictions based on this theory.

CHAPTER IV

A MODEL OF THE ELECTRON SPIN

1. Assumptions of the Model

I shall use the electron as an example of a spin-$\frac{1}{2}$ particle, but the following will apply equally well to other spin-$\frac{1}{2}$ particles. The model is based on the following assumptions:

(a) Each electron, at every given moment, has a definite spin in all directions, and its spin values are given by a Fermi function.

(b) All electron Fermi functions belong to a family of the form $F_0 = \{s_0 \circ \alpha \mid \alpha \in O_3\}$, for some fixed (yet unknown) Fermi function s_0.

(c) If s_1, s_2 are Fermi functions of two uncorrelated electrons then s_1, s_2 are spherically independent, that is, the satisfy eq. (3-8).

By 'uncorrelated pair of electrons' I mean a pair of electrons that are not interacting, or that were not interacting in the near past.

As a result of a (low energy) interaction of an electron with another particle or with a macroscopic apparatus the electron Fermi function s is transformed to another Fermi function s', but the transformation is always of the form $s' = s \circ \alpha$, for some $\alpha \in O_3$. I shall assume that in each such case the transformation α depends completely on the type of interaction and on the dynamic variables of the electron other then the spin itself. This assumption turns the model into a deterministic one.

We see immediately that the Operational Incommensurability Thesis is true in this model. Suppose that s is an electron Fermi function and suppose that we perform a measurement of the electron spin in the z direction. As a result of the interaction between the electron and the measurement apparatus, the electron spin function is transformed by an orthogonal transformation α, $s \to s \circ \alpha$. Let x be an arbitrary direction. Then we may have $s(x) \neq s(\alpha(x))$, in which case the measurement of the spin in the z direction has indeed altered the spin in the x direction.

2. Correspondence With Quantum Mechanics

In the formalism of quantum mechanics we associate with each spin-$\frac{1}{2}$ system a two-dimensional complex Hilbert space H_2. Let z be a fixed direction. Then the vectors $|+\rangle = \binom{1}{0}$ and $|-\rangle = \binom{0}{1}$ of H_2 represent the states spin 'up' and spin 'down' in the z direction, respectively.

Let (r,θ,ϕ) be a spherical coordinate system in physical space such that $z = (1,0,0)$ and let $x = (1,\theta,\phi)$ be an arbitrary unit vector in physical space. The orthogonal matrix

$$\alpha = \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \qquad (4\text{-}1)$$

transforms z to x, that is, $\alpha(z) = x$. This transformation is represented on H_2 by the unitary matrix:[1]

$$D_{1/2}(\alpha) = \begin{pmatrix} \cos(\frac{1}{2}\theta)e^{-i\frac{1}{2}\phi} & -\sin(\frac{1}{2}\theta)e^{-i\frac{1}{2}\phi} \\ \sin(\frac{1}{2}\theta)e^{i\frac{1}{2}\phi} & \cos(\frac{1}{2}\theta)e^{i\frac{1}{2}\phi} \end{pmatrix} \qquad (4\text{-}2)$$

Thus $D_{1/2}(\alpha)|+\rangle = \begin{pmatrix} \cos(\frac{1}{2}\theta)e^{-i\frac{1}{2}\phi} \\ \sin(\frac{1}{2}\theta)e^{i\frac{1}{2}\phi} \end{pmatrix}$ represents the state 'spin up in the x direction' and $D_{1/2}(\alpha)|-\rangle = \begin{pmatrix} -\sin(\frac{1}{2}\theta)e^{-i\frac{1}{2}\phi} \\ \cos(\frac{1}{2}\theta)e^{i\frac{1}{2}\phi} \end{pmatrix}$ is the state spin 'down' in the x direction. Note that $D_{1/2}(\alpha)|+\rangle$ is always a non-trivial superposition of $|+\rangle$ and $|-\rangle$ and the same is true for $D_{1/2}(\alpha)|-\rangle$. In the Copenhagen Interpretation this means that a particle which has a definite spin in the z direction does not have a definite spin value in any other directions x such that $x \neq \pm z$. Note also that each unit vector in the Hilbert space H_2 has a representation of the form $D_{1/2}(\alpha)|+\rangle$ or $D_{1/2}(\alpha)|-\rangle$ for some orthogonal matrix α. (This is not true in the three-dimensional and higher cases, a fact that causes some difficulties, as we shall see in the next chapter.)

In the present model the states $D_{1/2}(\alpha)|+\rangle$ $D_{1/2}(\alpha)|-\rangle$ have a different interpretation. Let s be the electron Fermi function. s assigns a definite spin value in every direction. Let z be fixed and let x be as before. Define the state $|s(x)\rangle \in H_2$ by:

$$|s(x)\rangle = D_{1/2}(\alpha)|+\rangle = \begin{pmatrix} \cos(\frac{1}{2}\theta)e^{-i\frac{1}{2}\phi} \\ \sin(\frac{1}{2}\theta)e^{i\frac{1}{2}\phi} \end{pmatrix} \quad \text{if } s(x) = +\frac{1}{2}$$

$$|s(x)\rangle = D_{1/2}(\alpha)|-\rangle = \begin{pmatrix} -\sin(\frac{1}{2}\theta)e^{-i\frac{1}{2}\phi} \\ \cos(\frac{1}{2}\theta)e^{i\frac{1}{2}\phi} \end{pmatrix} \quad \text{if } s(x) = -\frac{1}{2} \tag{4-3}$$

whenever $x = (1,\theta,\phi)$. Thus, in our model, the complete spin state of an electron whose Fermi function is s is given by the set of unit vectors $\{|s(x)\rangle \mid x \in S^{(2)}\} \subseteq H_2$. From (4-3) we get immediately

$$|\langle s(x)|s(z)\rangle|^2 = \begin{cases} \cos^2(\frac{1}{2}\theta) & \text{if } s(z) = s(x) \\ \sin^2(\frac{1}{2}\theta) & \text{if } s(z) \neq s(x) \end{cases} \tag{4-4}$$

Comparing with formula (3-6) we get the relation between the model and quantum mechanics:

$$|\langle s(x)|s(z)\rangle|^2 = p_{z\theta}[\{w \in c(z,\theta) \mid s(w) = s(x)\}] = \frac{1}{2} + 2s_1(z)s_2(z)\cos^2\theta \tag{4-5}$$

which means that $|\langle s(x)|s(z)\rangle|^2$ *is the conditional expectation of spin equals* $s(x)$ *on the circle* $c(z,\theta)$, for θ = arc cos $(x \cdot z)$. One of the nontrivial properties of Fermi functions is the fact that this conditional expectation is invariant under the interchange of the roles of x and z

$$|\langle s(x)|s(z)\rangle|^2 = p_{z\theta}[\{w \in c(z,\theta)|s(w) = s(x)\}] = p_{x\theta}[\{w \in c(x,\theta)|s(w) = s(z)\}] = |\langle s(z)|s(x)\rangle|^2.$$

Having established this correspondence I can generalize it to account for the relation between the quantum mechanical picture of composite systems of identical particles and the way my model handles such systems.

In quantum mechanics the states of a two-electron system of uncorrelated electrons are given by the vectors of the four-dimensional tensor product $H_2 \otimes H_2$. Again, let z be a fixed direction and let $|+\rangle_i$, $|-\rangle_i$ be the states spin 'up', spin 'down', respectively, in the z direction for the i^{th} electron (i = 1,2). The four four-dimensional vectors $|++\rangle$, $|+-\rangle$, $|-+\rangle$, $|++\rangle$ form a natural basis for $H_2 \otimes H_2$. The vector $|+-\rangle = |+\rangle_1 |-\rangle_2$, for example, corresponds to the state 'spin up in the z direction for the first electron and spin down in the z direction for the second electron'. The other vectors have a similar interpretation. Let s_1, s_2 be the two-electron Fermi functions. Since the electrons are uncorrelated I assume that s_1, s_2 are spherically independent. For $x = (1,\theta,\phi)$ let $|s_1(x)\rangle_1$, $|s_2(x)\rangle_2$ be the corresponding states. Then $|s_1(x)s_2(x)\rangle$ is the four-dimensional state 'spin $s_1(x)$ in the x direction for the first electron and spin $s_2(x)$ in the x direction for the second electron'. From the definition of tensor products we get:

$$|\langle s_1(x)s_2(x)|s_1(z)s_2(z)\rangle|^2 = |\langle s_1(x)|s_1(z)\rangle|^2|\langle s_2(x)|s_2(z)\rangle|^2 =$$
$$= p_{z\theta}[\{w \in c(z,\theta)|s_1(w) = s_1(x)\}] \times p_{z\theta}[\{w \in c(z,\theta)|s_2(w) = s_2(x)\}] \tag{4-6}$$

But since s_1, s_2 are spherically independent we have:

$$p_{z\theta}[\{w \in c(z\theta) | s_1(w) = s_1(x)\}] \times p_{z\theta}[\{w \in c(z,\theta) | s_2(w) = s_2(x)\}] =$$

$$= p_{z\theta}[\{w \in c(z,\theta) | s_1(w) = s_1(x)\} \cap \{w \in c(z,\theta) | s_2(w) = s_2(x)\}]. \quad (4\text{-}7)$$

Therefore, *taking the tensor product $H_2 \otimes H_2$ as the state space of a 2-electron system corresponds with my assumption that s_1, s_2 are spherically independent*, in the sense that $|\langle s_1(x)s_2(x)|s_1(z)s_2(z)\rangle|^2$ is indeed the conditional expectation of spin equals $s_1(x)$ for the first electron *and* spin equals $s_2(x)$ for the second electron on the circle $c(z,\theta)$, for $\theta = \text{arc}\cos(x \cdot z)$.

In the case that the electrons are correlated we cannot assume any longer that the electron spin functions are spherically independent. They are, in fact, spherically correlated in a way which is given by the singlet and triplet states. These correlations will be dealt with in chapter V.

3. <u>One Polarization</u>

Suppose that we are given a beam of electrons. If this beam has not been prepared in some special specified way we can assume that the electrons in the beam form a random sample. This means, in my model, that the electron Fermi functions are spherically independent. Thus if $s_1, s_2, s_3 \dots s_n$ are the electron Fermi functions, the sequence $s_1, s_2 \dots s_n \dots$ is assumed to be random, i.e., to satisfy eq. (3-8). Denote

$$A = \{w \in S^{(2)} \mid \frac{1}{n} \sum_{i=1}^{n} s_i(w) \xrightarrow[n\to\infty]{} 0\}.$$

Then from theorem (3-6) we know that $E(A \mid z, \frac{\pi}{2}) = 1$ for all z, and moreover if $z \in A$ we have $E(A \mid z) = 1$. In the extension of probability that I utilize this means that, for large n, the probability that $n^{-1} \sum_{i=1}^{n} s_i(w)$ is close to 0, is 1. This 'explains' the fact that when we polarize a random beam by a Stern-Gerlach magnet oriented in our arbitrary direction w the beam splits into two sub-beams of approximately the same intensity, namely half the beam with $s_i(w) = +\frac{1}{2}$ and half with $s_i(w) = -\frac{1}{2}$.

4. The Einstein-Podolsky-Rosen-Bohm Experiment

I shall assume in this section that the Fermi functions of two electrons in the singlet state are opposite. That is, if s,s* are the two Fermi functions we have $s(w) = -s^*(w)$ for all $w \in S^{(2)}$. This assumption is consistent with the results of the E.P.R. experiment when the magnets are oriented in the same direction, and we shall also see, in V, that it is consistent with the interpretation of the quantum mechanical expectation values as given by my model.

In the E.P.R. experiment we have a source that emits electron pairs in the singlet state and thus we have two beams that travel in opposite directions 'left' and 'right'. Let $s_1, s_2, \ldots s_n \ldots$ be the Fermi functions associated with the left beam, then $s_1^*, s_2^*, \ldots, s_n^* \ldots$ are the Fermi functions of the electrons associated with the right beam, and $s_i^*(w) = -s_i(w)$ for all $w \in S^{(2)}$ and $i = 1, 2, \ldots$. The left beam itself is composed of a random sample of electrons, which means that the electron Fermi functions in that beam are spherically independent, or in other words, $s_1, s_2, \ldots, s_n \ldots$ is a random sequence (satisfies formula 3-8).

Suppose that we put Stern-Gerlach magnets at equal distance on opposite sides of the source. The magnet on the left is oriented in the z direction and the magnet on the right oriented in the x direction. As a result of the interaction with the magnet the left beam splits into two sub-beams, $\{s_n\} = \{s'_n\} \cup \{s''_n\}$, such that $s'_n(z) = +\frac{1}{2}$ and $s''_n(z) = -\frac{1}{2}$. Moreover, since $\{s_n\}$ is a random sequence we know that the relative frequency of the subsequence $\{s'_n\}$ in the original sequence is $\frac{1}{2}$.

What is the relative frequency of 'spin up on the left and spin down on the right'? Since we have assumed that $s_i(w) = -s^*_i(w)$, this frequency equals the relative frequency of 'spin up in the z direction and spin up in the x direction' for the left beam, which is the relative frequency of spin up in the z direction (which is $\frac{1}{2}$) times the relative frequency of 'spin 'up' in the x direction' in the subsequence $\{s'_n\}$. Denote

$$B = \{w \in S^{(2)} \mid \frac{1}{k} \sum_{i=1}^{k} s'_i(w) \to \frac{w \cdot z}{2}\} .$$

Then $E(B \mid z) = 1$ (theorem (3-6)). Hence, with probability 1 the relative frequency of spin up in the x direction in the subsequence $\{s'_n\}$ is $\frac{1}{k} \sum_{i=1}^{k} [s'_i(x) + \frac{1}{2}] \to \frac{1}{2} + \frac{x \cdot z}{2}$. The probability in this case is given by the z-directional expectation $E(B \mid z)$. To sum up: the frequency of spin up on the left and spin down on the right is, most probably, $\frac{1}{2}(\frac{1}{2} + \frac{x \cdot z}{2}) = \frac{1}{2}\cos^2(\frac{1}{2}\theta)$, where θ is the angle between x and z.

Note that in calculating the probability of the 'event' B, I have used the z-directional expectation, utilizing the fact that z is the direction with which I have defined the subsequence s_i'. We could have interchanged the roles of z and x (that is, calculated the frequencies of 'spin down in the x direction and spin down in the z direction' on the right beam) without altering the result. The whole argument is based on the *validity* of the principle of locality. The one strange, non-classical, element is the definition of probability as a z-directional expectation. We have seen that $E(B \mid y) = 0$ for many directions $y \neq \pm z$ (theorem 3-7), but if we identify the probability of B with $E(B \mid z)$ the result comes out right. Thus, we have a local explanation of the E.P.R. experiment provided that we accept a concept of probability which depends on the particular directions in which the experiment is *actually* carried out.

5. <u>Double Polarization Experiment</u>

Consider the following experimental set-up. We pass an electron through a Stern-Gerlach magnet oriented in the z direction. If the electron goes 'up' we further polarize it by a second magnet oriented in the x direction (Fig. 4.1).

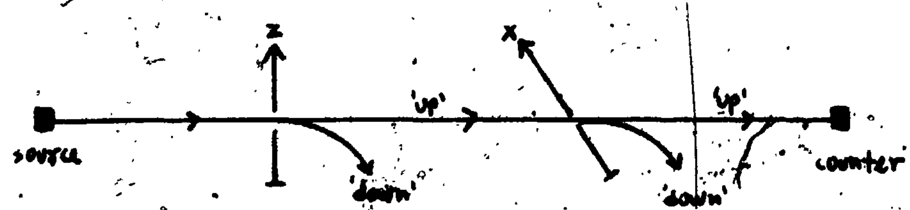


Fig. 4.1

In this way we can measure the frequency of 'spin up through the first magnet and spin up through the second magnet' in a random sample (a beam) of electrons. The frequency predicted by quantum mechanics and measured in experiments is again $\frac{1}{2}\cos^2(\frac{1}{2}\theta)$, where $\theta = \text{arc}\cos(x \cdot z)$. That is, the frequency in a double polarization experiment equals that of the E.P.R. experiment. This is a highly non-trivial observation, since the first measurement introduces a disturbance in the spin values, in the sense that the spin in the x direction of a single electron may flip as a result of the first measurement. The point is that the first measurement does not seem to introduce a *statistical* disturbance, that is, the statistical properties of the beam before the first measurement do not change during the measurement. In other words, *the expectation values are invariant*. This is precisely what my model predicts. Let s be an electron Fermi function before the first measurement and s' after the measurement. We have $s' = s \circ \alpha$ for $\alpha \in O_3$. In *any* such measurement the spin in the z direction itself does not change, that is, $s(z) = s'(z)$. (To see this, we can put $x = z$, i.e., polarize the beam twice in the z direction. We shall see that all the electrons that went 'up' through the first apparatus go 'up' through the second.) We can thus conclude that in this case we have $\alpha(z) = z$.

Let $s_1, s_2 \ldots s_n \ldots$ be the electron Fermi functions before the first measurement. Then $s_1 \circ \alpha_1, s_2 \circ \alpha_2 \ldots s_n \circ \alpha_n$ are the Fermi function after the measurement and $\alpha_i(z) = z$ for $i = 1, \ldots n$. If $s_1', s_2' \ldots s_k'$ is the subsequence of the original sequence of all those functions for which $s_i'(z) = +\frac{1}{2}$, then $s_1' \circ \alpha_1 \ldots s_k' \circ \alpha_k$ are the Fermi functions of the electrons in the beam that go up after the first measurement. Since $s_1; s_2 \ldots; s_n$ was random so is $s_1 \circ \alpha_1 \ldots s_n \circ \alpha_n$ and thus $E\left(\left\{\frac{1}{k} \sum_{i=1}^{k} s_i' \circ \alpha_i(w) \to \frac{w \cdot z}{2}\right\} \mid z\right) = 1$, which means that most probably $\frac{1}{2}(\frac{1}{2} + \frac{x \cdot z}{2}) = \frac{1}{2}\cos^2(\frac{1}{2}\theta)$ of the original beam will go up through both apparatuses.

6. Macroscopic Magnetism

Spin and angular momentum of charged particles are 'responsible' for magnetism. Thus, if an electron has 'spin up' in the z direction it carries magnetic moment $+\frac{1}{2}\mu_0$ in that direction, where μ_0 is the Bohr magneton. In my model, the electron has a definite spin in all directions and consequently we must conclude that it has definite magnetic moment $\mu_0 s(x)$ in each direction x (where s is the electron Fermi function). Hence microscopic magnetic moments are not vectors. Macroscopic magnetic moments on the other hand, are vectors and the reason for this is as follows: Let $s_1 \ldots s_n$ be the Fermi functions of a large sample of electrons all confined to a relatively small portion of space. Assume that all the electrons have spin up in the z direction. In that case, the total magnetic moment of the sample in the z direction is

$$\mu_z = \mu_0 \sum_{i=1}^{n} s_i(z) = \frac{\mu_0 n}{2}.$$

If n is large enough, μ_z is of macroscopic dimensions. Let x be an arbitrary direction. Then the magnetic moment of the sample in the x direction is:

$$\mu_x = \mu_0 \sum_{i=1}^{n} s_i(x) = \frac{\mu_0 n}{2}(x \cdot z) + \mu_0 \gamma_n(x)$$

where $\gamma_n(x) = \sum_{i=1}^{n} s_i(x) - \frac{n(x \cdot z)}{2}$. Since $s_i(z) = +\frac{1}{2}$ for all $i = 1 \ldots n$ and since $s_1 \ldots s_n$ are spherically independent we have:

$$E\left(\left\{\frac{1}{n} \sum_{i=1}^{n} s_i(w) \xrightarrow[n\to\infty]{} \frac{z \cdot w}{2}\right\} \mid z\right) = 1.$$

Thus, again, if we identify 'probability' with the z-directional expectation we conclude that for large n, with probability 1, the quantity $\mu_0\gamma_n(x)$ represents a diminishing percentage of the magnetic moment μ_x. That is, for large n:

$$\mu_x \approx \frac{n(x \cdot z)}{2} = \mu_z(x \cdot z) = \mu_z \cos \theta$$

where θ is the angle between x and z.

In other words: *The vector-like behaviour of macroscopic magnetic moments is a statistical phenomenon* in my model.

One can imagine, at this stage, what the world would have been like if the distribution of spin values over $S^{(2)}$ had not been given by a Fermi function, but rather by another type of spin function. In that case, Bell's inequality might not have been violated (which is good) but then, also, macroscopic magnetic moments would not have behaved like vectors (which is bad).

This observation, incidentally, applies not just in the framework of my model. The quantum mechanical expectation values that are 'responsible' for the fact that macroscopic magnetic moments are vectors are precisely those that violate Bell's inequality. Thus, in some sense, the existence of macroscopic magnetic vector fields is a large scale manifestation of a violation of Bell's inequality.

7. Principles for Testing the Model

A random sample of electrons is described by a random sequence of Fermi functions. The model, on the other hand, is assumed to be deterministic, that is, one can calculate, in principle, the exact transformation $s \to s \circ \alpha$ which the electron Fermi function undergoes during an interaction. Using such a calculation one can design a sample of electrons in which randomness has been destroyed. As a result the sample will manifest statistical behaviour which deviates from the predictions of quantum mechanics.

Suppose that we pass an electron through a Stern-Gerlach apparatus oriented in the z-direction. Let s be the electron Fermi function before it enters the apparatus. After it leaves the apparatus the electron Fermi function is $s \circ \alpha$ for $\alpha \in O_3$. We know that in *every* such experiment the electron spin in the z direction does not change. Thus we can assume that in this case α leaves z invariant $\alpha(z) = z$. The orthogonal transformation α depends on the dynamic variables of the electron other than the spin itself and on the properties of the magnetic field of the apparatus. I do not have a theory of the nature of this dependence but in the following I shall assume that such a theory is available.

Consider the following thought experiment: Take an electron and pass it through a Stern-Gerlach apparatus in the z direction. Suppose it goes 'up'. Now further polarize it in the x direction for x *orthogonal to* z. Suppose it goes 'up' again. Finally polarize it a third time in the z direction. Let s be the original Fermi function of the electron. It undergoes three orthogonal transformations

$$s \to s \circ \alpha_1 \to s \circ (\alpha_2\alpha_1) \to s \circ (\alpha_3\alpha_2\alpha_1)$$

where $\alpha_1(z) = \alpha_3(z) = z$ and $\alpha_2(x) = x$. If we can control the dynamic variables and magnetic fields so that α_2 is a 180° rotation about the x-axis then *the electron will definitely go 'down' through the third apparatus*. This occurs since

$$\alpha_3\alpha_2\alpha_1(z) = \alpha_3\alpha_2(z) = \alpha_3(-z) = -z$$

and hence $s(\alpha_3\alpha_2\alpha_1 z) = s(-z) = -s(z)$. Quantum mechanics as well as my model predicts that in a *random* sample of electrons about $\frac{1}{8}$ of the electrons will go 'up' through all three apparata. Thus we have a thought experiment which exhibits about 12% deviation from the predictions of quantum mechanics. The thought experiment is not intended, of course, as an instruction for the experimentalist. It indicates, I believe, that any theory of the nature of the transformations $s \to s \circ \alpha$ could be tested in principle. I believe that such a theory could be developed in a relativistic framework. The reason for this belief lies in the phenomenon called 'Wigner Rotation'.[1] If a particle is in the eigenstate 'spin 'up' in the z direction' in the laboratory frame of reference, the same particle would appear to a moving observer to be in a spin eigenstate for another direction x. In the language of my model this turns out to be equivalent to the statement that the electron Fermi function has been rotated. This Wigner Rotation may have to do with the above transformation, $s \to s \circ \alpha$, and the argument for this is by analogy with another case:

In a given frame of reference the energy of a freely moving particle is composed of its mass and its kinetic energy. Another observer will detect the same total amount of energy, but the composition of this amount into mass and kinetic energy will be different for him. This, formally speaking, is the source of the mass-energy equivalence. It is a highly non-trivial fact that one can transform mass into kinetic energy in the *same* frame of reference. This indeed could be done only in the presence of a field. By analogy, the presence of a field may induce a Wigner Rotation in a fixed frame of reference and this is precisely the rotation that we are looking for.

CHAPTER V

MASSIVE SPIN-1 PARTICLES AND CORRELATED SYSTEMS

1. <u>Bose Functions</u>

By analogy to the spin $-\frac{1}{2}$ case, I shall introduce the concept of a Bose function which assigns a definite spin value in all directions of space:

<u>Definition (5-1)</u>: A Bose function is a function $j : S^{(2)} \to \{-1,0,1\}$ such that, for all $z \in S^{(2)}$ and all $0 < \theta < \pi$,

$$E(j \mid z,\theta) = j(z) \cos \theta \tag{5-1}$$

$$E(j^2 \mid z,\theta) = \sin^2\theta + j^2(z) \frac{1}{2}(3 \cos^2 \theta - 1). \tag{5-2}$$

Before dealing with the proof of the fact that such functions exist I shall estabilsh some of their properties. First note that both j and j^2 are totally spherically integrable. A simple calculation shows

$$E(j) = \frac{1}{2} \int_0^\pi E(j \mid z,\theta) \sin \theta \, d\theta = 0 \tag{5-3}$$

$$E(j^2) = \frac{1}{2} \int_0^\pi E(j^2 \mid z,\theta) \sin \theta \, d\theta = \frac{2}{3} \tag{5-4}$$

Denote $A_+ = \{w \in S^2 \mid j(w) = +1\}$, $A_0 = \{w \in S^{(2)} \mid j(w) = 0\}$, and $A_- = \{w \in S^{(2)} \mid j(w) = -1\}$. Then we have:

$$j = \chi_{A_+} - \chi_{A_-} \qquad j^2 = \chi_{A_+} + \chi_{A_-} = 1 - \chi_{A_0}. \tag{5-5}$$

Thus, using (5-1), (5-2), and (5-5) we can calculate the conditional expectation of 'spin up', 'spin zero', 'spin down' on the circle $c(z,\theta)$. These are given in the following table

	$E(A_+ \mid z,\theta)$	$E(A_0 \mid z,\theta)$	$E(A_- \mid z,\theta)$
$j(z) = 1$	$\cos^4(\frac{1}{2}\theta)$	$\frac{1}{2}\sin^2\theta$	$\sin^4(\frac{1}{2}\theta)$
$j(z) = 0$	$\frac{1}{2}\sin^2\theta$	$\cos^2\theta$	$\frac{1}{2}\sin^2\theta$
$j(z) = -1$	$\sin^4(\frac{1}{2}\theta)$	$\frac{1}{2}\sin^2\theta$	$\cos^4(\frac{1}{2}\theta)$

(5-6)

The conditional expectations in formula (5-1), (5-2) are rotationally invariant, that is, for all $\alpha \in 0_3$, all $z \in S^{(2)}$, and $0 < \theta < \pi$:

$$E(j \circ \alpha \mid z,\theta) = j(\alpha(z)) \cos\theta \tag{5-7}$$

$$E(j^2 \circ \alpha \mid z,\theta) = \sin^2\theta + j^2(\alpha(z)) \tfrac{1}{2}(3\cos^2\theta - 1). \tag{5-8}$$

The proof of this fact is along the same lines as the proof of lemma (3-4). I shall now prove:

<u>Theorem (5-1)</u>: *There exists a Bose function.*

<u>Proof</u>: For $0 < \theta \le \frac{\pi}{2}$, put $R(z,\theta) = c(z,\theta) \cup c(-z,\theta) \cup \{-z,z\}$ and $K = \{R(z,\theta) \mid 0 < \theta \le \frac{\pi}{2},\ z \in S^{(2)}\}$. Again, there is a well-ordering of K which is isomorphic with Ω. Define j by induction on the order. At each stage λ one splits the circle $c(z_\lambda,\theta_\lambda)$ into *three* disjoint subsets whose measures are given in table (5-6): If $j(z_\lambda)$ has not yet been defined or if $j(z_\lambda) = 1$ the measures are given by the first row of the table, if $j(z_\lambda) = 0$ by the second row, and if $j(z_\lambda) = -1$ by the third row. By M.C.H. this procedure is possible at any stage , since

at any such stage j is already defined only on a subset of measure zero of $c(z_\lambda, \theta_\lambda)$. Q.E.D.

By analogy to the spin $-\frac{1}{2}$ case we have

Definition (5-2): 1) Two Bose functions are spherically independent if for all $0 < \theta \leq \pi$ and $z \in S^{(2)}$:

$$E(j_1 j_2 \mid z,\theta) = E(j_1 \mid z,\theta) E(j_2 \mid z,\theta)$$
$$E(j_1^2 j_2^2 \mid z,\theta) = E(j_1^2 \mid z,\theta) E(j_2^2 \mid z,\theta) \tag{5-9}$$

2) A sequence $j_1, j_2 \ldots j_n \ldots$ of Bose functions is random if: for all $z \in S^{(2)}$ (except maybe finitely many) and all $0 < \theta < \pi$

$$E(j_{i_1} \ldots j_{i_k} \mid z,\theta) = \prod_{\ell=1}^{k} E(j_{i_\ell} \mid z,\theta)$$
$$E(j_{i_1}^2 \ldots j_{i_k}^2 \mid z,\theta) = \prod_{\ell=1}^{k} E(j_{i_\ell}^2 \mid z,\theta) \tag{5-10}$$

for all $k \geq 2$ and choices of indices $1 \leq i_1 \quad i_2 < \ldots < i_k \leq \infty$.

By the same technique as in theorem (3-5) we can prove that random sequences of Bose functions exist and, moreover, that there is a random sequence of the form $\{j \circ \alpha^n\}_{n=1}^{\infty}$, where j is fixed and $\alpha \in O_3$. The law of large numbers when applied to Bose functions is:

Theorem (5-2): 1) *Let $j_1, j_2 \ldots j_n \ldots$ be a random sequence of Bose functions all satisfying $j_i(z_0) = j_1(z_0)$. Then*

$$E(\{w \in S^{(2)} \mid \frac{1}{n} \sum_{i=1}^{n} j_i(w) \xrightarrow[n\to\infty]{} j_1(z_0)(w \cdot z_0)\} \mid z_0) = 1. \tag{5-11}$$

2) *Let $j_1 \ldots j_n \ldots$ be a random sequence of Bose functions. Then*

$E(\{w \in S^{(2)} \mid \frac{1}{n} \sum_{i=1}^{n} j_i(w) \xrightarrow[n\to\infty]{} 0\} \mid z, \frac{\pi}{2}) = 1$ *for all* $z \in S^{(2)}$ *(except maybe finitely many). Moreover, if* $\frac{1}{n} \sum_{i=1}^{n} j_i(z) \xrightarrow[n\to\infty]{} 0$ *then*

$E(\{w \in S^{(2)} \mid \frac{1}{n} \sum_{i=1}^{n} j_i(w) \to 0\} \mid z) = 1.$

The proof is identical to that of theorem (3-6).

2. A Model for Massive Spin-1 Particles

The assumptions for this model are similar to those of the spin-$\frac{1}{2}$ case:

a) Each massive, spin-1 particle at every given moment has a definite spin in all directions and its spin values are given by a Bose function.

b) All the Bose functions of a particular species of spin-1 particles have the form $F_0 = \{j_0 \circ \alpha \mid \alpha \in O_3\}$ for some fixed, yet unknown, Bose function j_0.

c) If j_1, j_2 are the Bose functions of two identical uncorrelated particles, then j_1, j_2 are spherically independent.

A low energy interaction will transform the Bose function j by an orthogonal transformation $j \to j \circ \alpha$. Again, I assume that α is uniquely determined by the initial conditions and the type of interaction, an assumption that makes the model deterministic.

To establish the correspondence relations between this model and the usual formalism of quantum mechanics, let (r,θ,ϕ) be a polar coordinate system in physical space and let $z = (1,0,0)$. If $x = (1,\theta,\phi)$ is an arbitrary unit vector, the matrix α in formula (4-1) transforms z to x. This matrix has the following unitary representation on the three-dimensional Hilbert space:

$$D_1(\alpha) = \begin{pmatrix} \cos^2(\frac{1}{2}\theta)\, e^{-i\phi} & -\frac{1}{\sqrt{2}} \sin\theta\, e^{-i\phi} & \sin^2(\frac{1}{2}\theta)\, e^{-i\phi} \\ \frac{1}{\sqrt{2}} \sin\theta & \cos\theta & -\frac{1}{\sqrt{2}} \sin\theta \\ \sin^2(\frac{1}{2}\theta)\, e^{i\phi} & \frac{1}{\sqrt{2}} \sin\theta\, e^{i\phi} & \cos^2(\frac{1}{2}\theta)\, e^{i\phi} \end{pmatrix} \tag{5-12}$$

Let j be a Bose function and let $|j(x)>$ be the first, second, or third column of the matrix $D_1(\alpha)$ in the case $j(x) = 1$, $j(x) = 0$, $j(x) = -1$, respectively. Then we have

$$|<j(x) \,|\, j(z)>|^2 = p_{z\theta}[\{w \in c(z,\theta) \,|\, j(w) = j(x)\}] \tag{5-13}$$

that is, $|<j(x) \,|\, j(z)>|^2$ is the conditional expectation of 'spin equals $j(x)$' on $c(z,\theta)$. This could easily be verified from table (5-6). The rest of the analysis of massive spin-1 particles follows exactly the same pattern as the spin-$\frac{1}{2}$ case. In particular, the expectation values in (5-6), in conjunction with the law of large numbers, theorem (5-2), explain various polarization phenomena as well as the behaviour of magnets.

3. Kochen and Specker Theorems

Unlike the spin-$\frac{1}{2}$ case, the present situation involves further complications. I have assumed that the states of a massive spin-1 particle are given by a Bose function j, and consequently by the three-dimensional complex vectors $|j(x)>$. By definition, each such vector is a column of a matrix of the form $D_1(\alpha)$, for some α. There are, however, many three-dimensional unit vectors that do not have this representation, e.g., $|\psi\rangle = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} |+\rangle + \frac{1}{2} |0\rangle + \frac{1}{2} e^{i\frac{\pi}{2}} |-\rangle$, and from the principle of superposition we know that each such vector represents a physical state of a massive spin-1 particle. Thus, quantum mechanics is much richer in states than my model.

This observation is related to a theorem by Kochen and Specker.[1] Let H_3 be the three-dimensional complex Hilbert space and let L_3 be the ortholattices of subspaces of H_3. A truth-function on L_3 is a function $h: L_3 \to \{0,1\}$ which satisfies the following conditions:

1) If e_1, e_2, e_3 is an orthogonal triple in L_3, then for one and only one of these vectors we have $h(e_i) = 1$.

2) If $a < e \vee e'$, $e \perp e'$ and $h(e) = h(e') = 0$, then $h(a) = 0$.

Kochen and Specker proved that there are no truth-functions on L_3, and in fact there is a finite sublattice of L_3 on which no truth function exists. This means, in particular, that L_3 cannot be imbedded in a Boolean algebra.

Consider now the family of vectors which do have a representation as one of the columns of the matrix $D_1(\alpha)$, for some α of the form (4-1). Call this family Δ. It is easy to see that *there are* truth functions on Δ. To establish that, let j be a Bose function and for all $x \in S^{(2)}$ identify $|j(x)\rangle$ with the (one dimensional) subspace spanned by $|j(x)\rangle$. Now put $h(|j(x)\rangle) = 1$ for $x \in S^{(2)}$ and $h = 0$ on $\Delta \backslash \{|j(x)\rangle \mid x \in S^{(2)}\}$. h is clearly a truth function wich satisfies the above two conditions on the atoms of the set Δ. In fact this procedure gives us sufficiently many truth functions to separate the points of Δ (this fact is related to theorem 2.1) and thus Δ is imbeddable in a Boolean algebra.[3]

Having made this observation, we can conclude that the difficulty pointed out by Kochen and Specker results from the principle of superposition, that is, from the assumption that every unit vector of H_3 represents a physical state of a *single* spin-1 particle. In order to circumvent this difficulty we can thus introduce the following:

<u>Superselection Rule</u>: *The states realizable by a single massive spin-1 particle are all of the form* $|j(x)>$, *where* j *is a Bose function and* $x \in S^{(2)}$.

What about the rest of the unit vectors in H_3? These represent some *statistical information* about a particle that *belongs to a collection* which has some distinguished properties. Thus, for example, the vector

$$|\psi> = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} |+> + \frac{1}{2} |0> + \frac{1}{2} e^{i\frac{\pi}{2}} |-> \tag{5-14}$$

represents the following information about a massive spin-1 particle:

(i) It belongs to a collection in which the relative frequency of spin 'up', 'zero', and 'down' in the z-direction are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$, respectively.

(ii) The collection is polarized in such a way that it has no particle with spin zero in the $x = (0,\theta,\phi)$ direction for $\phi = 0$ and $\theta = \text{arc tan}\ (\sqrt{2})$.

In particular (ii) follows, since the state 'spin zero in the x direction' for $x = (1,\theta,\phi)$ is given by the second column of the matrix $D_1(\alpha)$, that is, $|0,x> = -\frac{1}{\sqrt{2}} \sin\theta\ e^{-i\phi} |+> + \cos\theta\ |\theta> + \frac{1}{\sqrt{2}} \sin\theta\ e^{i\phi} |->$ and thus for $\phi = 0$ and $\theta = \text{arc tan}(\sqrt{2})$ we get $<\psi|0,x> = 0$.

It is easy to see that every unit vector of H_3 can be interpreted in a similar way. Note that in the present model a particle in the state $|\psi\rangle$ is assumed to have a definite spin in all directions and the information given by $|\psi\rangle$ merely represents our ignorance of these values.

Kochen and Specker's theorem also applies in the case of the lattice of subspaces of the three-dimensional Euclidean (real) space, L_3'. That is, there are no truth functions on L_3'. This fact also causes some complications of a different kind.

Let x,y,z be three orthogonal directions in real space. If J_x, J_y, J_z are the angular momentum (spin) operators in these directions then the square angular momentum (spin) operator $J^2 = J_x^2 + J_y^2 + J_z^2$ has eigenvalues of the form $\ell(\ell+1)$, where ℓ is an integer or half-integer. In the case of a free massive spin-1 particle we have $\ell = 1$, and thus the eigenvalue of J is $\ell(\ell+1) = 2$. The problem is how to interpret this fact in a model which assigns a definite spin j(x) in every direction x. A natural answer seems to be that we impose the condition:

$$j^2(x) + j^2(y) + j^2(z) = 2, \tag{5-15}$$

for every orthogonal triple x,y,z and every function j which assigns the appropriate spin values. Condition (5-15) means that for each orthogonal triple x,y,z one and only one direction has spin zero. Suppose that this is the case and define a truth function h on L_3' by $h(x) = 1$ iff $j(x) = 0$ and $h(x) = 0$ otherwise. (We identify every one-dimensional subspace with a unit vector that spans it.) Then h is a truth function on L_3'. But we know that truth functions do not exist on L_3', hence no function j can satisfy (5-15). Indeed, Bose functions do not satisfy (5-15), but rather a weaker condition. Let z be a direction such that $j(z) = 0$. Then, from table (5-6) we get $E(A_0 \mid z, \frac{\pi}{2}) = \cos^2(\frac{\pi}{2}) = 0$, ~~that is, the~~ expectation of 'spin 0' on the plane orthogonal to z is 0, and $E(A_+ \mid z, \frac{\pi}{2}) = E(A_- \mid z, \frac{\pi}{2}) = \frac{1}{2}$. On the other hand, if $|j(z)| = 1$ we have $E(A_+ \mid z, \frac{\pi}{2}) = E(A_- \mid z, \frac{\pi}{2}) = \frac{1}{4}$ and $E(A_0 \mid z, \frac{\pi}{2}) = \frac{1}{2}$. Thus we can expect that for some orthogonal triples x,y,z we shall have $j^2(x) + j^2(y) + j^2(z) \neq 2$.

The operator J^2 corresponds in my model to a measurement of the average of j^2 over $S^{(2)}$. For any orthogonal triple x,y,z (or any triple whatsoever) we have from (5-2)

$$E(j^2 \mid x) + E(j^2 \mid y) + E(j^2 \mid z) = 3\,E(j^2) = 3\times\frac{2}{3} = 2. \qquad (5\text{-}16)$$

The same interpretation applies in the two-dimensional spin - $\frac{1}{2}$ case. In that case, $\ell = \frac{1}{2}$ and thus $\ell(\ell+1) = \frac{3}{4}$. Indeed, if s is a Fermi function we have $s^2 \equiv \frac{1}{4}$, and thus $3E(s^2) = \frac{3}{4}$.

4. Interacting Spin - $\frac{1}{2}$ Systems

The Fermi functions of two interacting electrons or two electrons that have interacted in the near past cannot be taken as spherically independent. If s_1, s_2 are the two Fermi functions of the electrons after the interaction we shall have $E(s_1 s_2 \mid z, \theta) \neq E(s_1 \mid z, \theta)E(s_2 \mid z, \theta)$, which means that s_1, s_2 are *spherically correlated*. Not any correlation of this kind can be realized by a pair of electrons; rather only two types of correlation exist. This fact results from Pauli's exclusion principle, which states (in this case) that the composite state of the two-electron system is either the singlet state or the triplet state. Pauli's exclusion principle is, in fact, a superselection rule which makes certain unit vectors in the four-dimensional Hilbert space $H_2 \otimes H_2$ void of any physical meaning. We shall see that the Pauli's exclusion principle amounts in the present framework to the following statement:

<u>The Correlation Principle</u>: Let s_1, s_2 be the Fermi functions of two electrons. Then just one of the three cases applies:

1) s_1, s_2 are spherically independent, in which case

$$E(s_1 s_2 \mid z, \theta) = s_1(z) s_2(z) \cos^2\theta. \tag{5-17}$$

2) s_1, s_2 are in the triplet state, in which case

$$E(s_1 s_2 \mid z, \theta) = \frac{\sin^2\theta}{8} + s_1(z) s_2(z) \frac{1}{2}(3\cos^2\theta - 1) \tag{5-18}$$

3) s_1, s_2 are in the singlet state, in which case

$$E(s_1 s_2 \mid z, \theta) = -\frac{1}{4}. \tag{5-19}$$

We have immediately:

<u>Lemma (5-3)</u>: *Let* s_1, s_2 *be two electron Fermi functions and put* $j = s_1 + s_2$. *Then*

1) $E(j \mid z, \theta) = j(z) \cos\theta$ *for all* $z \in S^{(2)}$, $0 < \theta < \pi$;

2) s_1, s_2 *are in the triplet state iff* j *is a Bose function*;

3) s_1, s_2 *are in the singlet state iff* $E(j^2 \mid z, \theta) \equiv 0$.

The proof is trivial. In particular 2) follows from the fact that

$j^2 = (s_1 + s_2)^2 = \frac{1}{2} + 2s_1s_2$, and thus (5-18) is satisfied iff formula (5-2) is satisfied for $j = s_1 + s_2$.

Note that both in the singlet and triplet states the product s_1s_2 is totally spherically integrable and we have $E(s_1s_2) = \frac{1}{12}$ in the triplet state and $E(s_1s_2) = -\frac{1}{4}$ in the singlet state. In the case of uncorrelated electrons we have $E(s_1s_2 \mid z) = \frac{1}{3}\, s_1(z)s_2(z)$, and in this case s_1s_2 is not totally spherically integrable.

To see how the correlation principle corresponds with Palui's exclusion principle, let again (r,θ,ϕ) be a polar coordinate system. Let $z = (1,0,0)$ and let

$$|{+}{+}\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad |{+}{-}\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad |{-}{+}\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad |{-}{-}\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

correspond with the four states of the two-electron spins in the z direction. Thus, for example, $|{-}{+}\rangle$ is the state spin 'down' for the first electron and spin 'up' for the second electron in the z direction. The matrix α of (4-1) is represented on $H_2 \otimes H_2$ by:

$$D_{\frac{1}{2}\times\frac{1}{2}}(\alpha) = \begin{pmatrix} \cos^2(\frac{1}{2}\theta)\, e^{-i\phi} & -\frac{\sin\theta}{2}\, e^{-i\phi} & -\frac{\sin\theta}{2}\, e^{-i\phi} & \sin^2(\frac{1}{2}\theta)\, e^{-i\phi} \\ \frac{\sin\theta}{2} & \cos^2(\frac{1}{2}\theta) & -\sin^2(\frac{1}{2}\theta) & -\frac{\sin\theta}{2} \\ \frac{\sin\theta}{2} & -\sin^2(\frac{1}{2}\theta) & \cos^2(\frac{1}{2}\theta) & -\frac{\sin\theta}{2} \\ \sin^2(\frac{1}{2}\theta)\, e^{i\phi} & \frac{\sin\theta}{2}\, e^{i\phi} & \frac{\sin\theta}{2}\, e^{i\phi} & \cos^2(\frac{1}{2}\theta)\, e^{i\phi} \end{pmatrix} \tag{5-20}$$

Suppose that s_1, s_2 are two spin functions and let $|s_1(x)\rangle\ |s_2(x)\rangle$ be the states given by formula (4-3) for the functions s_1, s_2. Then $|s_1(x)s_2(x)\rangle = |s_1(x)\rangle|s_2(x)\rangle$ is the first, second, third, or fourth column of $D_{\frac{1}{2}\times\frac{1}{2}}(\alpha)$ in the case $s_1(x) = s_2(x) = \frac{1}{2}$, $s_1(x) = -s_2(x) = \frac{1}{2}$, $s_1(x) = -s_2(x) = -\frac{1}{2}$, and $s_1(x) = s_2(x) = -\frac{1}{2}$, respectively. Thus, in all cases we have:

$$|\langle s_1(x)s_2(x)|s_1(z)s_2(z)\rangle|^2 = p_{z\theta}[\{w \in c(z,\theta)|s_1(w) = s_1(x)\}]p_{z\theta}[\{w \in c(z,\theta)|s_2(w) = s(x)\}].$$

And we have seen that in the case s_1, s_2 are spherically independent:

$$|\langle s_1(x)s_2(x)|s_1(z)s_2(z)\rangle|^2 = p_{z\theta}[\{w \in c(z,\theta)|s_1(w) = s_1(x) \text{ and } s_2(w) = s_2(x)\}].$$

In order to establish the correlation of the singlet and triplet states, note that the representation $D_{\frac{1}{2}\times\frac{1}{2}}(\alpha)$ in (5-20) is reducible.

Let

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (5\text{-}21).$$

U is a unitary matrix that transforms the vectors of the basis $|++\rangle$, $|+-\rangle$, $|-+\rangle$, $|--\rangle$ to those of $\frac{|+-\rangle-|-+\rangle}{\sqrt{2}}$, $|++\rangle$, $\frac{|+-\rangle+|-+\rangle}{\sqrt{2}}$, $|--\rangle$, respectively. We have:

$$U^{-1}D_{\frac{1}{2}\times\frac{1}{2}}(\alpha)\,U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2(\frac{\theta}{2})e^{-i\phi} & -\frac{1}{\sqrt{2}}\sin\theta\, e^{-i\phi} & \sin^2(\frac{\theta}{2})e^{-i\phi} \\ 0 & \frac{1}{\sqrt{2}}\sin\theta & \cos\theta & -\frac{1}{\sqrt{2}}\sin\theta \\ 0 & \sin^2(\frac{\theta}{2})e^{i\phi} & \frac{\sin\theta}{\sqrt{2}}e^{i\phi} & \cos^2(\frac{\theta}{2})e^{i\phi} \end{pmatrix} \quad (5\text{-}22)$$

Or, in other words, $U^{-1}D_{\frac{1}{2}\times\frac{1}{2}}(\alpha)\,U = \begin{pmatrix} 1 & 0 \\ 0 & D_1(\alpha) \end{pmatrix}$, where $D_1(\alpha)$ is the matrix given by (5-12). Let s_1, s_2 be the Fermi function of two correlated electrons. By the Pauli exclusion principle we know that the coupled system is either in the singlet state or in the triplet state. Thus, for $x = (1,\theta,\phi)$ put $j(x) = s_1(x) + s_2(x)$, and define $|j(x)\rangle$ to be the first column of (5-22) in case the electrons are in the singlet state, and $|j(x)\rangle$ to be the second, third, or fourth columns of (5-22) in case the electrons are in the triplet state and $j(x) = 1, 0, -1$, respectively. Consider the singlet state. Using the correspondence between the present model and quantum mechanics we have in this case for all x, z and $0 < \theta < \pi$:

$$|\langle j(z)|j(x)\rangle|^2 = p_{z\theta}[\{w \in c(z,\theta)\,|\,j(w) = j(x)\}] = 1. \qquad (5\text{-}23)$$

Suppose that for some x we have $j(x) = s_1(x) + s_2(x) \neq 0$, say $j(x) = 1$. Then from (5-23) we get that for all $z \in S^{(2)}$ and $0 < \theta < \pi$ $E(\{w \in S^{(2)}\ |s_1(w) + s_2(w) = 1\}|\ z,\theta) = 1$. But this is a contradiction, since $E(\{s_1(w) + s_2(w) = 1\}\,|\,z,\theta) = E(\{s_1(w) + s_2(w) = -1\}\,|\,-z,\theta)$ while $\{s_1(w) + s_2(w) = 1\} \cap \{s_1(w) + s_2(w) = -1\} = \emptyset$. Hence, $j(x) \neq 1$ for all x. From symmetry it follows that $j(x) \neq -1$ for all x and hence $j(x) = 0$ for all x, which means that $s_1(x) = -s_2(x)$ for all $x \in S^{(2)}$. In the triplet state we have, from the correspondence formula (5-13), $|\langle j(z)|j(x)\rangle|^2 = p_{z\theta}[\{w \in c(z,\theta)\,|\,j(w) = j(x)\}]$ and these expectation values are exactly those given by table (5-6), that is, $j = s_1 + s_2$ is a Bose function.

5. A Note on Zero Mass, Spin-1 Particles

For such particles the distribution of spins satisfies a different expectation formula: Let $e : S^{(2)} \to \{-1,1\}$. We shall say that e is an Einstein function if for all w, z, and $0 < \theta < \pi$:

$$e(-w) = e(w) \tag{5-24}$$

$$E(e \mid z,\theta) = e(z)\cos(2\theta). \tag{5-25}$$

Put $E_+ = \{w \in S^{(2)} \mid e(w) = +1\}$. Then $\chi_{E_+} = \frac{1}{2}(e+1)$ and thus

$$E(E_+ \mid z,\theta) = \frac{1 + e(z)\cos(2\theta)}{2} = \begin{cases} \cos^2\theta & e(z) = 1 \\ \sin^2\theta & e(z) = -1 \end{cases} \tag{5-25}$$

By the same technique as before we can show that Einstein functions exist. Note that in this case we assume that the spin in opposite directions *is the same* (formula (5-24)), unlike the case of massive particles.

If we assume that photons are associated with Einstein functions in an analogous way that electrons are associated with Fermi functions we can explain polarization phenomena with light. (The law of large numbers is slightly different in this case. Thus if $e_1, e_2, \ldots, e_n, \ldots$ is a random sequence of Einstein functions we have

$$E\left(\{w \in S^{(2)} \mid \frac{1}{n}\sum_{i=1}^{n} e_i(w) \to 0\} \mid z, \frac{\pi}{4}\right) = 1 \text{ for all } z \in S^{(2)} \text{ (except}$$

finitely many), and if $\frac{1}{n}\sum_{i=1}^{n} e_i(z) \to 0$ then

$$E\left(\{w \in S^{(2)} \mid \frac{1}{n}\sum_{i=1}^{n} e_i(w) \to 0\} \mid z\right) = 1.$$ Also, if all the e_i's satisfy

$e_i(z_0) = +1$ for some z_0, then

$E(\{w \in S^{(2)} \mid \frac{1}{n}\sum_{i=1}^{n} e_i(w) \to (2(w \cdot z_0)^2 - 1)\} \mid z_0) = 1)$. Note also that Einstein functions are not totally spherically integrable since $E(E_+ \mid z) = \frac{1}{3}$ when $e(z) = +1$ and $\frac{2}{3}$ when $e(z) = -1$. This seemingly disadvantageous property of Einstein functions might turn into an advantage when we consider electromagnetic interactions. Two random electrons start an interaction with spherically independent spin functions s_1, s_2. In this case, $E((s_1 + s_2)^2 \mid z) = \frac{1}{3}$ when $s_1(z) \neq s_2(z)$ and $\frac{2}{3}$ when $s_1(z) = s_2(z)$. The interaction ends in the singlet or triplet state and in both cases, if s_1', s_2' are the Fermi functions after the interaction, the function $(s_1' + s_2')^2$ is totally spherically measurable. Since the interaction occurs via an interchange of (virtual) photons we cannot expect that Einstein functions will turn out to be spherically integrable. The particular values of the directional expectations in this case indicate that the present model might be consistent with those aspects of quantum electrodynamics that are related to the spin.

In the modern approach to elementary particles the prevailing theory asserts that all elementary particles are compositions of 'truely' elementary constituents which are either spin - $\frac{1}{2}$ massive particles (electrons, muons, quarks) or (with the exception of the graviton) massless spin-1 particles (photons, neutrinos, gluons). Thus, the spin of every particle is calculated from the spins of its elementary constituents using the addition rules of angular momentum. In the present model, this calculation procedure can be accounted for by the spherical correlations among the spin functions of the constituents, following the same line of thought as in the example of two correlated spin - $\frac{1}{2}$ particles.

CHAPTER VI

PHILOSOPHICAL REMARKS

1. Physics and Realism in Mathematics

'The essence of mathematics lies in its freedom' said Georg Cantor. 'Freedom', for him, meant the open opportunity to exercise one's imagination and conceive of mathematical objects which are not constructible by finitistic means. His dictum was, in fact, a reply to those mathematicians (Kronecker in particular) who accused him of playing with abstract nonsense.[1]

Set theory (or 'Cantor's paradise', as Hilbert called it) is not any more a questionable section of mathematics; on the contrary, it is conceived by most mathematicians as a cornerstone, if not the most fundamental theory of all. Still, those sections of set theory which involve highly non-constructive principles are suspicious and mathematicians are advised not to utilize them whenever possible.

Perhaps the prime example of a non-constructive procedure is conveyed by the principle of choice.[2] The axiom of choice (A.C.) states the following:

Given a family F *of non-empty mutually disjoint sets, there exists a set* A *which is composed of precisely one element from each set of* F.

This axiom does not specify a rule or an algorithm by which one can construct the set A. Sometimes such a rule exists (in which case we do not need the axiom in the first place) and sometimes not. A beautiful demonstration of these two cases is given by the following example due to Bertrand Russell: If F is an infinite family of pairs of shoes then a 'canonical' algorithm exists for constructing A (take the left shoe of every pair). If, on the other hand, F is an infinite family of pairs of socks no such algorithm seems to exist.

K. Gödel and P. Cohen[3] have respectively proved the consistency and independence of the axiom of choice relative to Zermelo-Fraenkel set theory (Z.F.). Thus the axiom of choice can be added or dropped from set theory or can be replaced by a weaker principle (such as the axiom of countable choice). It is an interesting problem to determine which portions of mathematics, and in particular mathematical analysis, could be reconstructed without the axiom of choice. Some answers are available. For example, without the axiom of choice (at least some weaker version of it) one cannot prove that a denumerable union of denumerable sets is denumerable (that is, $\aleph_0\aleph_0 = \aleph_0$). In fact, there is a model of Z.F. + $\neg$A.C. in which the set of reals is a countable union of countable sets. It goes without saying that very little analysis can be recovered in such a model and measure theory, in particular, takes a very strange form. Another example: there is a model of Z.F. + $\neg$A.C. in which one can prove the existence of an infinite set of reals that has no denumerable subset. As a result, it follows that in this model there is a subset S of reals and a number r such that r is in the topological closure of S while there is no sequence $\{x_n\} \subseteq S$ such that $x_n \to r$. In this and in similar models the topological and sequential definitions of continuity do not coincide.

In short, it is obvious that (some version) of the axiom of choice is indispensable if we wish to reconstruct classical analysis. The principle of choice, on the other hand, has some consequences which are considered as unfortunate, the prime example being the existence of nonmeasurable sets. Lusin was the first to 'construct' a nonmeasurable subset of the reals, but the most striking example is the so called 'Banach-Tarski paradox' which was formulated in 1924 and which states the following:

'Using the axiom of choice one can cut a ball into finitely many pieces which can be so rearranged that one obtains two balls of the same size as the original one'.

The 'pieces' into which the ball is decomposed are nonmeasurable sets to which one cannot assign numbers that indicate their volume without violating the additivity or invariance of 'volume'. Such counterintuitive examples led people to suspect the validity of the full-fledged axiom of choice. In 1970 Solovey[4] constructed a model of Z.F. + $\neg$A.C. in which a weak principle of choice applies and in which every subset of a Lebesgue space is measurable. Solovey's model is also very attractive from a topological standpoint and its only disadvantage is that it depends on a set-theoretical axiom which asserts the existence of a large cardinal, an axiom that has far less intuitive appeal than the axiom of choice itself.

In any case, from a purely formal standpoint there seem to be no grounds for rejecting or accepting any of these models. Once basic set theory (that is Z.F.) is taken as a legitimate part of mathematics, there is no formal criterion which decides the truthfulness or falsity of any of its models.[5] In spite of this, judgments as to the truthfulness or plausibility of models of the reals are made (unconsciously) almost daily and bizarre models of the continuum, such as those I have mentioned above, are rejected as 'counterexamples' or 'pathologies'. The principles that guide such decisions are indeed not part of formalized mathematics but they are nevertheless certainly an essential part of the enterprise of mathematics, since they serve to distinguish between 'normal' mainstream analysis and esoteric research. I believe that the main guideline for making decisions with respect to models of the reals is physical-geometrical intuition. In some cases, those which involve 'higher' set theoretical properties, no physical guideline exists and intuition becomes a matter of idiosyncracy. Thus P. Cohen (naturally) argued that the continuum hypothesis is false[6] and K. Gödel indicated that he believes that the axiom of constructibility is false.[7]

Both these axioms have no immediate connection with the physical world and are hardly, if at all, used in analysis. Most mathematicians therefore reserve judgment with respect to their truth or falsity. The more important cases, however, are those which involve principles that bear immediate significance for standard analysis. I believe that there exists a basic intuition with respect to some topological and measure-theoretic properties of the reals which is shared by most mathematicians and whose source is ultimately physical reality.

In historical perspective it seems self-evident that the origin of a large portion of mathematics, and certainly mathematical analysis, is basic intuition with respect to space, time, and motion. It is no accident that the infinitesimal calculus was invented hand-in-hand with classical mechanics. Both Newton and Leibniz did not even make the distinction between 'pure' and 'applied' mathematics. I have already mentioned the confidence of D'Alembert and his colleagues in the validity of analysis in spite of apparent logical paradoxes. This confidence stemmed from physical reality. D'Alembert did not need a logical demonstration of the consistency of the calculus. Its truthfulness was self evident. Analysis at that period represented more than a *model* of reality, it was reality.

The confidence in the self-evidence of mathematics was shaken only in the nineteenth century and resulted from two developments, firstly the invention of non-Euclidean geometry and secondly the research into the foundations of analysis.

With non-Euclidean geometry one had, for the first time, an example of a mathematical construction which is perfectly consistent and which defies pre-analytic intuitions with respect to space. Of course, such models could have been dismissed as abstract games but they were not. Gauss himself conducted experiments with light rays trying to decide whether space is flat or has some small curvature. He concluded that on the basis of the data he collected this question could not be settled.[8] As far as I know this was the first incident in the history of physics where people came to suspect that their mathematical theories are representations or models of reality and alternative explanations might exist. Empirical data in itself does not necessitate a unique model.

The work on the foundations of analysis, particularily by Bolzano and later on Weierstrass had a somewhat different effect. Bolzano constructed such 'unphysical' objects as a continuous function which is nowhere differentiable and thus demonstrated that physical intuition alone falls far short of capturing the vast number of mathematical objects which can be generated by one's imagination. Mathematical analysis and mathematical physics became two divorced enterprises; the later invokes some principles of the former but cannot ultimately justify it. Indeed, these two historical developments led to the invention of set theory (which was motivated initially by problems in analysis, mainly Fourier analysis) and of mathematical logic (the first precise formalization of a mathematical theory was Hilbert's 'Grundlagen der Geometrie').

Nobody seriously doubts the tremendous importance of mathematical logic and set theory for mathematics and for domains outside mathematics but their development, I believe, has also had some unfortunate consequences for the philosophy of mathematics. There is, of course, a vast number of approaches to the foundations of mathematics and I shall concentrate on two examples: Intuitionism and Logicism.[9] In spite of their obvious differences, both these schools share (with many other approaches) the basic belief that the continuum can be reduced to the descrete. Both the intutionist and logicist start with a collection of finite objects whose existence is self-evident. For the former these are natural numbers generated by the successor operation, assumed to be (psychologically) a priori; for the logicist these are finite classes of objects satisfying trivial predicates. From these 'self-evident' atoms the intuitionist and logicist proceed to construct increasingly complex and abstract objects and ultimately real numbers. They differ, of course, in what they take as a 'legitimate construction procedure' and a 'legitimate existence proof', and as a result they end up with a different continuum. Thus, the intuitionist class (or what they call, 'species') of reals is not even linearly ordered, while the logicists' construction yields infinitely many models, some of them truely bizarre from a topological perspective.

These attempts to reconstruct the continuum from descrete objects rest on different metaphysical premises. What is forgotten or ignored in this construction process is the historical and intuitive conception of the continuum that bears some features which are part of what generations of mathematicians have taken as self-evident. This is particularily true with respect to the intuitionist, and it is indeed an irony that a school which bears such a name denies the existence of a linear order among the reals, on the basis that such an order relation is 'nonconstructible'.

Geometrical intuition, or rather the human conceptualization of physical space and the flux of time, are not less basic than natural numbers, the successor operation, or finite classes of objects. There is strong empirical evidence from psychology and physiology which indicates that some 'continuous' mental operations such as translation and rotation of space, or the formation of a three-dimensional picture from a pair of 'flat' ones, are elementary to the extent that they may have an analogue on the level of the hardware of the brain itself.[10] If this is the case, one can argue against the intutionistic conception of the continuum on the same metaphysical grounds that motivate the intuitionist in the first place. In the same way, one can argue against the formalist who allows for 'freely generated' topologically bizarre models of the reals or, for that matter, against any attempt to reduce the continuum to 'more elementary' objects.

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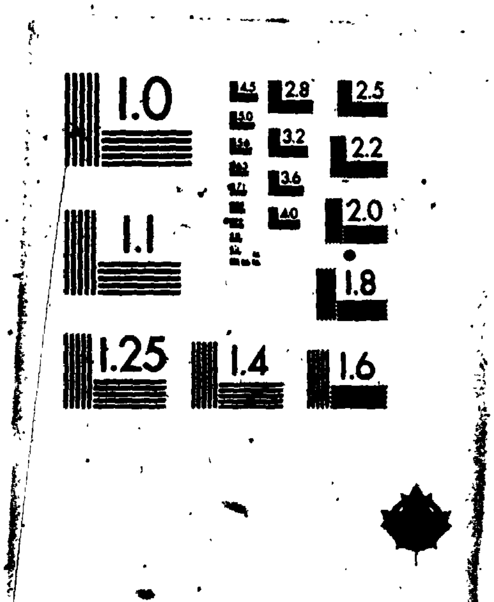

The psychologically a priori[11] human conception of the continuum may serve as a guideline for rejecting some consistent models of the reals as false, but it falls far short of being complete. A vast number of questions remain unsettled, obvious examples being the continuum hypothesis or the weaker measure theoretic continuum hypothesis which I introduced in section III.

The fact that there exists a language which allows for the formulation of such axioms resulted from the divorce between pure and applied mathematics. This divorce, in turn, gave rise to a dialectical process where certain theorems which were proved in pure mathematics were fed back into physics. As a result, increasingly abstract mathematical objects are assumed to have reference in nature.

The prime example of such a process is again non-Euclidean geometry where Riemannian manifolds model space-time. Another example is the theory of Hilbert spaces, or functional analysis in general, which forms the mathematical basis of quantum mechanics. The spectral theorem, which is a cornerstone of Von Newmann's axiomatization of quantum theory, is proved with the aid of the maximal ideal theorem, a non-constructive principle closely related though somewhat weaker than the axiom of choice, and with the aid of Tychonoff's lemma, which is strictly equivalent to the axiom of choice.[12] A third example is measure theory, invented by Lebesgue and Borel at the turn of the century[13] and introduced to physics through the work of N. Wiener in his classical account of Brownian motion and stochastic processes in general.[14] One of the results of Wiener's work, incidentally, was the demonstration that continuous, nowhere-differentiable curves (those 'unphysical' objects) are essential for the understanding of such physical processes.

Mathematical physics, in short, applies increasingly abstract mathematical theories in an attempt to understand Nature. In doing so it assumes the validity of mathematical principles that bear no immediate intuitive relation to observable facts. This process calls for a metaphysical justification: what is the relation between mathematical models and the reality they allegedly depict? There are a variety of positions regarding this problem. Most fall between two extremes, the instrumentalist and extreme realist.

Upon accepting a physical theory, the realist commits himself to the existence of the mathematical objects involved. Thus, accepting Wiener's theory of Brownian motion as true involves an ontological commitment with respect to measurable subsets of physical space. It is assumed, in other words, that these sets exist in physical reality and bear physical significance. This kind of commitment involves some complications, because accepting the reality of some abstract sets forces the realist to decide their abstract (set-theoretical) properties. For the realist, the question whether there is a subset of points *in physical space* whose cardinality is strictly between $\aleph_0$ and $2^{\aleph_0}$ is perfectly legitimate, and the answer is either yes or no. The fact that the continuum hypothesis does not seem to bear any relation to observed physical phenomena does not reduce the legitimacy of the question: if sets are real so are their properties.

This type of consideration is the source of the model-theoretic argument against realism.[15] In the absence of both formal and physical criteria which decide some of the properties of 'real' sets it is not clear anymore in what sense they are real. It is true that one should not confuse what there is with one's knowledge of what there is but, nevertheless, if one's knowledge is doomed to be partial in principle, ontological commitments seem arbitrary and ungrounded.

For the instrumentalist, an application of even the most abstruse piece of mathematics in physics is justified provided that the resulting theory is efficient in interpolating observational data, predicting new phenomena, or at least instrumental in clarifying and simplifying the logical network of theoretical physics. Such an application does not involve ontological commitment, the only a priori constraint is consistency. Questions such as the validity of the continuum hypothesis can be dismissed as irrelevant since abstract set-theoretical properties play no part in the application of mathematics as an instrument of physics.

I believe that the instrumentalist will find it hard to explain the overwhelming success of science, especially physics. For him this success seems accidental, a stroke of luck. To understand why this is the case one can twist the model-theoretic argument and apply it against instrumentalism. As we have seen, there is a large variety of models of the reals, some of them truly bizarre in their topological and measure-theoretic features. If mathematical objects in a physical theory do not refer, but rather serve as instruments, it seems a particularily fortunate historical accident that mathematical physicists picked up the class of models which share what we call 'reasonable' topological properties, and that these properties 'accidentally' turned out to be 'good instruments' for predicting the behaviour of physical objects. Of course, a much more plausible explanation of the historical development of physics is one which assumes at least some sense of realism with respect to the mathematical objects involved in analysis. An outline of such an explanation was given previously.

Between the realist and instrumentalist extremes, I find myself closer to the former. Even though it is impossible, in principle, to decide all the relations among mathematical objects which appear in a physical theory, I still think that even abstract statements carry definite truth value. The scientific enterprise is, in part, a process of eliminating false models. Being a realist does not entail the belief that this process converges to one single theory, 'The Truth', only that it progresses to the point where nature, and our limited intellectual and sensory resources for understanding nature, find their equilibrium.

No one can tell in advance where this 'point of equilibrium' is to be found. In particular, there is no a priori way of deciding which of the abstract properties involved in pure mathematics bears physical significance.

Nonmeasurable sets may serve as an illustration of this point. The existence of such objects is consistent only when the full-fledged axiom of choice is assumed to be valid. In spite of this, such sets may turn out to have physical significance. Some people have indicated that nonmeasurable sets are 'unphysical'.[16] Such a statement is certainly unjustified from an instrumentalist standpoint, nor is it justified from a realistic point or view. Indeed, the latter have to be more cautious with respect to abstract objects since their application entails ontological commitments. Nevertheless, for the realist, mathematical physics comes as a package deal. The proof that commuting families of self-adjoint operators have spectral resolution involves the axiom of choice. This is considered a legitimate use of the axiom which yields a desirable consequence for quantum theory and as such bears immediately on our understanding of microphysics. The proof that nonmeasurable sets exist invokes the same principle of choice but the result is 'unfortunate' and 'unphysical'. What is the basis for making a distinction between these two cases? The answer is that there is no such *a priori* basis, save perhaps aesthetic judgment (in the best case) or prejudice.

In the model that I have proposed, certain types of nonmeasurable sets are instrumental and their use may even result in predictions. Taken as a realistic model it involves the assumption that nonmeasurable subsets of directions form part of physical reality. In such a realistic framework the proof of Bell and others that no classical joint distribution exists for the spin correlations amounts to a demonstration that the joint distribution must be non-classical, that is, nonmeasurable. Thus, if quantum statistics can be consistently recovered in a theoretical framework that invokes nonmeasurable sets, then *within such a framework*, the violation of Bell's inequality by observed frequencies serves as *evidence* that nonmeasurable subsets of physical space exist. The argument for this follows closely the case of relativity. The path of light rays near the sun serves as a piece of evidence that space-time is not flat. As in other cases, this particular phenomenon may have an explanation which lies outside the domain of general relativity.[17] This fact, however, does not reduce the status of the Eddington experiment as a triumph of relativity, even though it is only within the framework of relativity itself that the connection between light paths and space-time geometry is made.

It goes without saying that relativity gained its status on the basis of a variety of observations, not just one phenomenon. In principle, however, the analogy with nonmeasurable sets is clear. Even though the E.P.R. experiment may have numerous different explanations and even though the connection between observed frequencies and non-measurable distributions becomes clear only within the model itself, this experiment nevertheless, may be taken as evidence for the model. Of course, further evidence and theoretical work are essential for making a final decision among the large number of alternative interpretations of quantum mechanics. The problem of interpretation is not entirely metaphysical (see section VI.3 below).

The abstract statement 'nonmeasurable sets exist' may thus turn out to be empirically decidable in the same way that statements about space-time geometry are empirically decidable. The same may turn out to be true with respect to other set-theoretical statements. There is no a priori basis for ruling out the (admittedly remote) possibility that even the continuum hypothesis bears physical significance.

2. Quantum Mechanics and the Foundations of Probability

Measure theory has grown out of geometrical intuition, measure being the generalization of concepts such as 'length', 'area', 'volume', etc. The pioneering work of Wiener, Kolmogorov, and von Neumann established the affinity between measure and probability. This affinity stems from two sources. The first is purely formal: normalized measures satisfy the same set of requirements that one would expect probability distributions to follow; the second is intuitive: in certain cases probabilities do have an immediate geometrical counterpart.

Thus, if A, B, $A \subseteq B$, are two bounded sets in the Euclidean plane, say, the probability that a point picked up at random from B lies in A is the relative area of A in B. These observations led to the so called axiomatic or formal approach to probability where this theory is conceived as a part of the broader measure theory.

Probability theory, which forms the basis of statistical inference, is however not merely a branch of mathematics. Statistical inference is involved practically in all empirical enterprises, from physics through weather predictions to sociology and even the study of history. This wide range of applications calls for a justification. Why are the axioms of probability, and consequently statistical inference, valid? Again, there is a wide variety of answers and most fall between the two extremes: the frequentist approach whose modern formulator was von Mises,[18] and the subjectivist approach due to Keynes.[19] I shall not attempt to survey and criticize these different approaches but rather concentrate on a more general problem. Any thinker who takes issue with probability theory, regardless of his particular approach, should be able to address the following question: Can you conceive of a hypothetical situation where the axioms of probability and, consequently, statistical inference are observed to be invalidated? If the answer is 'yes' then the theory of probability is empirical, that is, the application of the classical form of statistical inference should be justified in every case separately since the particular classical form is theory-dependent and not a priori. If the answer to the above question is in the negative, then the theory of probability is a priori and statistical inference takes the same classical form all across the board. I suspect that most probabilists share the latter view, namely that statistical inference is metatheoretical and follows the same pattern in every case. This statement must be qualified when we consider von Mises. He explicitly took statistics to be a branch of the empirical sciences. This, however, did not shatter his conviction that statistical inference is universally valid (provided of course that the assumptions of independence and randomness, which are made in every particular application, are justified on independent empirical grounds).

Immanuel Kant was the philosopher who introduced the famous distinction between analytic and synthetic a priori statements. Analytic propositions are tautologies, self-evident truths which nobody can deny: 'black is black' or 'no unmarried man is married'. As such, analytic statements do not convey any information about the world and their truthfulness follows from the structure of language. Synthetic a priori propositions, on the other hand, do convey non-trivial information about reality (or, for Kant, about the way we understand reality), but nevertheless can never be falsified, that is, one cannot conceive of a situation where they go wrong. One of Kant's famous examples of such a class of statements was the axioms of (Euclidean) geometry. As it turned out, this choice was particularily unfortunate and a constant source of attack on Kant's distinction in general.

I know of no similar explicit statement with respect to the axioms of probability, but there is strong implicit evidence which suggests that a large majority of probabilists do take the axioms of probability to be synthetic a priori truths. One significant exception was Carnap[20] who took probability theory to be analytic. For him, the justification of its axioms follows the same line of reasoning as the justification of logical inference. The axioms of probability, however, are far from being void of information and they are certainly not implied, in any reasonable sense, by the axioms of logic. This is particularily true when continuously infinite event spaces are concerned, that is, in all applications of statistics in physics. But even in discrete cases, even when coin tosses are concerned, probability theory falls far short of being a tautology. This is evident when we consider the concept of 'randomness' which underlies all applications of statistics. The term 'random' is somewhat misleading since randomness, when mathematically formulated, is a highly structured concept that follows definite patterns. To put it bluntly: the (mathematical) concept of randomness conveys a definite form of organization. Thus assumptions of randomness, which clarify the relation between expectation values and relative frequencies, convey highly non-trivial information about the world, its repetitive nature and the similarity of distinct events. As a matter of fact, even in classical probability theory there is a variety of definitions of what 'random' means and different definitions are applied to different situations. Maybe there is a non-classical pattern of 'randomness', a new concept of 'chaos', one which gives rise to limiting frequencies that violate the axioms of classical probability? This is, I believe, a natural question to ask, in particular given the bizarre behaviour of microscopic particles. The fact that this possibility was by and large ignored is yet another demonstration of the magnitude of the conviction of probablists in the universal validity of their pattern of reasoning. This attitude is particularly striking where the logical positivists are concerned. For the positivist, statistical inference transcends its role as a tool in the sciences and takes the form of 'inductive logic' which underlies the very justification of science itself. As such, statistical inference can hardly be claimed to be valid on empirical grounds, for this claim immediately leads to circularity. If the justification of inductive inference is not empirical, and if the axioms of probability are not mere tautologies, then what are they if not 'synthetic a priori'? The irony is that one of the first goals of logical positivism, a goal to which a high priority was given, was to demonstrate that Kant's category of 'synthetic a priori statements' is void and Kant's distinction meaningless.[21] The 'synthetic a priori' has crawled, as it were, through the back door. It seems reasonable to assume that Carnap's insistence on the tautological status of probability stemmed from his recognition of this irony. Quantum mechanics, I believe, at least suggests the possibility that the axioms of probability, and consequently classical statistical inference, are not universally valid. At any rate, the findings of microphysics certainly undermine the justification of both the frequentist and the subjectivist for the universal validity of probability theory.

The frequencies measured in the E.P.R. experiment violate the axiom of additivity of probabilities and thus pose a problem for the 'objectivist'. Of course, one can always resort to a non-local explanation and maintain that measurements performed in two different space locations are not independent, in which case the mystery is 'explained'. The point is that *the only source of evidence for a breakdown of locality is the violation of additivity (or the correlation 'above threshold level') itself.* Any non-local theory is thus *based* on the belief that probability theory and, in particular, the concept of statistical independence, a priori fits all physical situations. The only way the frequentist can save his form of justification is, therefore, to resort to an explanation which is ad hoc and highly bizarre from a physical standpoint.

A similar difficulty faces the subjectivist. The subjectivist typically invokes a 'Duch Book' argument to justify his (subjective?) belief in the axioms of probability. Suppose that an agent consistently violates one of these axioms. In this case, one can design a gambling situation (sell the agent a "Duch Book') such that the poor fellow is bound to lose (in the long run) if he gambles according to his rules (as opposed to those of classical probability).

The E.P.R. experiment provides a beautiful parody on the Duch Book argument. Suppose that a (rational) agent is informed about the results of a spin measurement on the left hand side of an E.P.R. experimental set-up and consequently is asked to bet on the result of a right hand side spin measurement. Assume, also, that he is told that the results are always opposite when the magnets on both sides are aligned. Moreover, suppose that (as in the Newcomb paradox) God appears to the agent and assures him that no information of any form whatsoever travels between the magnets. Being a rational person, the agent derives Bell's inequality. This does not tell him how to bet, rather how *not to*, but as it turns out he is bound to lose money in the long run. Having had his pockets emptied, the agent may conclude that he was deceived by God, a rather blasphemous position which rests entirely on his belief in probability, or otherwise he may reconsider his own beliefs and his own concepts.

The E.P.R. experiment, and in fact the entire body of 'interference phenomena', strongly suggest that such a revision in concepts should be made. It is still an open problem to determine the precise nature of the alternative 'non-classical' probability theory and various suggestions are available.[22] My own model suggests one such possibility which has the advantage of being 'quasi-classical', the relation between expectation and frequencies is recovered by an appropriate concept of 'randomness' and by a law of large numbers.

3. The Status of Quantum Mechanics

"Our statements about the external world," said Quine,[23] "face the tribunal of sense experience not individually but only as a corporate body." This famous Duhem-Quine thesis entails, in particular, that when 'sense experience' clashes with existing theories and beliefs there is no unique way to adjust the theory so as to accommodate the new data. As a matter of fact, every portion of our beliefs may be transformed in the face of new evidence and this includes even logic, let alone the theory of probability.

The problem, of course, is to determine which is the adequate transformation. While there seems to be no recipe for scientific revolutions, no one principle that governs all scientific change, a reasonable guideline does exist, namely the principle of conservatism. Thus, even though economists often face large-scale falsifications of their macrotheories, one would hardly take seriously the suggestion that the principles of arithmetic are at the root of the difficulties and the rules of accounting should be appropriately adjusted.

The variety of microscopic phenomena was a significant historical example of a body of 'sense experience' that defied existing theories. It forced a significant change in belief, the nature of which is not yet completely understood. I do not think that one can eliminate the revolution that quantum mechanics has brought about, but I believe it is important to admit that the precise character and magnitude of this revolution is still unclear. To a certain extent microphysics is still in its revolutionary phase. This claim is controversial; most physicists take quantum mechanics as an established part of 'normal science'. On a deep level, however, physicists do not understand the nature of their calculations: "nobody understands quantum mechanics". What is usually referred to as 'the problem of the interpretation of quantum mechanics' is, in fact, the problem of determining the precise character of the Quinian transformation that quantum mechanics forces on us. The very fact that a vast number of different interpretations exist is in itself a triumph of the Duhem-Quine thesis and perhaps an indication that the seeds of the next revolution in physics lie in this problem.

The easiest and, in my opinion, most unsatisfactory way of understanding the nature of quantum mechanics is by purely metaphysical deliberations. There is practically no bound on the flexibility of natural language for inventing metaphorical concepts which bear on one's feelings of uneasiness and uncertainty and transform them into statements which appear to carry metaphysical weight. While such an exercise may restore one's confidence and sense of control it has, I believe, very little value beyond causing these purely psychological effects. Concepts such as 'complementarity', 'wave-particle dualism' and the like, have played the important historical role of allowing physicists to ignore fundamental issues at a time of rapid development of quantum theory, but they do not carry any explanatory power. At present, when quantum field theory has almost achieved its goal of formulating a unified model of elementary interactions, a growing feeling of uneasiness surfaces again. Beginning in the sixties, a vast body of literature on the meaning of quantum mechanics, ranging from alternative technical formulations to bizarre mystical speculations, emerged, and is gradually making its impact on main-stream physics.[24] A growing number of physicists find the 'official' Copenhagen interpretation unsatisfactory.

Another purely metaphysical approach which is rooted in a long and distinguished philosophical tradition, is that of the antirealist: Since the birth of modern physics (and even before) people have debated whether gravitational fields, say, are 'real' or else mere instruments, 'theoretical objects' that 'save the phenomena'. Gravitational fields, after all, ~~are~~ never directly observed; it is only their effects which are felt. Such an argument may be invoked against any physical model because 'theoretical objects', those which are never observed directly, are an essential part of any non-trivial theory. In classical physics, however, one could at least pretend to have a real model but, the antirealist argues, this is not the case with quantum mechanics. In a series of publications B. van Fraassen[25] has argued that the violation of Bell's inequality is in itself a strong indication that no reasonable realistic interpretation of quantum mechanics exists and, in particular, one cannot consistently hold to the principle of common cause.

Every argument from physics to metaphysics is risky. A physical phenomenon in itself does not entail any particular metaphysical view. Van Fraassen himself admits that a realist, though non-local, interpretation of quantum mechanics is still possible. One can add to this correct observation the possibility of a Copenhagen type of view which restores some sense of realism. Both these alternatives, the anti-realist may argue, are extremely ad hoc; the former involves obscure and not yet developed physical principles, and the latter metaphysical unclarity. Thus, on grounds of plausibility, it may be argued that the anti-realist approach is the only possible reasonable way of accounting for the relation between quantum theory and observed reality. Elementary particles and mathematical relations among their properties are instruments that 'save the phenomena' and assist in predicting new phenomena, but one cannot consistently maintain that they exist. They are, as it were, theoretical objects.

Van Fraassen's approach is a mixture of empiricism and instrumentalism. He does not take a completely skeptical approach and deny the reality of medium sized everyday objects (chairs, rocks, and the like). For him the level of confidence with respect to ontological commitments decreases with the physical size of the object and practically diminishes when elementary particles are concerned. On this level ontological commitments cease to be consistent.

When the size of an object becomes smaller our very observation of it involves a 'background theory'. Our claim to have determined the properties of microbes rests on the 'background theory' of the microscope, that is optics. As the size of the object decreases further; the background theory involved in our interpretation of the results becomes more complex. When elementary particles are concerned, the theory of observation is no longer in the background. Quantum mechanics is a theory of observation as well as the theory of that which is observed.

All this is true, but one essential question still remains: What is it that is being observed? Before the invention of the electron microscope, viruses were never observed. In spite of this, there existed an elaborate theory of viruses, in particular their heredity properties together with a classification of their mutations. This theory was based on the examination of the impact that these creatures left on populations of microbes. In fact, the only evidence that viruses were living creatures was based on the observation that their heredity properties seemed to follow a Mendelian statistical pattern, an extremely indirect inference.[26] This example serves to demonstrate that the technical or even principal constraints on observation have very little to do with the ontological status of the observed. It is not in the nature of a theoretical object to leave a trace in a bubble chamber. Something leaves a trace in the bubble chamber, effects have causes. In order to sense the absurdity of the alternative position consider the following Kafkaesque situations: A veteran uranium miner, inflicted with cancer, sues the mining company. His lawyer naturally argues that his client has been exposed to unacceptable levels of radiation, a fact that caused his cancer; the miner is therefore eligible for compensation. Upon hearing this surprising line of reasoning the attorney for the defence calls an anti-realist metaphysician as a principal witness. The latter, using diagrams and elaborate arguments (including Bell's) demonstrates that one cannot consistently maintain that the radiation did indeed cause the injury, since nobody can consistently hold to the view that particles, those alleged objects that transmit what is generally referred to as 'radiation,' exist beyond the theoretical formulation. In the cross-examination a natural question is asked: "Do you say, in other words, that the well-established correlation between levels of exposure and the frequency of cancer is accidental?" "I did not say that," answers the metaphysician, "I only said that one cannot consistently maintain that the radiation caused the disease".

What this little macabre tale shows is that the denial of the principle of causation is the kind of view which only a philosopher can entertain seriously. Wittgenstein's warning against such positions still stands.[27]

I believe that a proper understanding of microphenomena, while no doubt involving metaphysical commitments, is not an issue of metaphysics per se. As in the case of the virus research, we are now in the position of making statistical inference on the basis of highly indirect observations. Unlike the virus case, we cannot expect that better equipment will solve the problem completely. The point is that we are most likely far from understanding the complete set of interrelations that exist among observables. The problem of interpretation is thus, in part, a problem of the physical theory itself.

Still, the revolution of microphysics is not likely to disappear as a result of future developments in physics. The nature of this revolution will become clearer only when we have a better grasp of the particular physical principle that underlies the conceptual change. The example of relativity theory is instructive in this context. Relativity theory has revolutionized our concepts of space-time, mass, energy, etc. This conceptual transformation goes hand-in-hand with a clear and precise *physical* principle which underlies the theory, namely the constancy of the velocity of light. Once this physical principle is accepted, which is a big step, metaphysical and conceptual problems fall into perspective. In particular the transformation from classical mechanics to relativity does not involve a mysterious 'quantum leap'. By contrast, no such clear physical principle underlies quantum mechanics. The obvious candidate, the principle of uncertainty (or the commutation relations between observables) falls far short of explaining the enigma.[28] The other principle on which quantum mechanics is based, the principle of superposition is, firstly, not universally valid and, secondly and more importantly, refers to the theoretical construct of 'state' (a vector in a Hilbert space), a concept that bears no immediate observable consequences. As such, the principle of superposition is more the source of mystery then its explanation.

In the search for a physical mechanism that will serve as an explanation for the deviations of quantum physics from the classical picture, authors have incorporated into physics domains that have traditionally been considered independent of physical results. This, in conjunction with the Duhem-Quine thesis, is the source of 'quantum logic'. For the quantum logician, the principles of the propositional calculus are not a priori but rather empirical. Thus, the fact that 'A and B' is true iff A is true and B is true is not taken as an a priori valid statement, nor as a part of the definition of the connective 'and'. Rather, when A and B are statements about physical properties of a system, a different 'operational' definition of the connective 'and' applies. We can determine by two separate measurements whether

A is true and whether B is true and we can (sometimes) determine by a single measurement whether 'A and B' is true. The fact that these two occasions coincide is, so the quantum logician maintains, not a priori but rather empirical. It reflects certain properties of the observed system and the process of observation itself. Thus there may exist a situation where the rules of propositional calculus are violated. Reichenbach was the first to make such a suggestion in the context of quantum mechanics,[29] but the logical form that is usually referred to as 'quantum logic' is the one abstracted by Birkhoff and Von Neumann[30] from the structure of the lattice of closed subspaces of a Hilbert space. The principle whose violation carries the weight of the micro-physical revolution is, according to quantum logic, the axiom of distributivity. When this principle is not assumed to be valid inter-ference phenomena can be 'explained' in the same way that the constancy of the velocity of light is 'responsible' for the relativity of time.

The single most important reason for rejecting or at least remaining agnostic with respect to quantum logic, is that it is an extremely radical view. From a Quinian perspective no body of knowledge, logic included, is immune to change in the face of new and surprising 'sense experience'. A transformation of propositional logic is, how-ever, so radical a move that it causes a necessary adjustment of prac-tically all our beliefs. Thus, for example, it entails that the arith-metic of natural numbers is sometimes false. Two plus two may not be four, in spite of Wittgenstein's claim that to deny that 2 + 2 = 4 is a 'misunderstanding'.[31] It is a tautological fact that with an adequate change in the rules of arithmetic one can devise a 'book-keeping' mechanism that 'explains' why interference occurs. An appeal to such a radical explanation can be made only as a last resort. It seems to me much more advisable to try to understand why two plus two sometimes *appears* to be different from four and the answer for this, I believe, may lie with some 'paradoxical' features of the continuum (see for example the Banach-Tarski paradox).

Regardless of my own idiosyncratic views, one may argue on the basis of conservatism that it is the theory of probability, not logic, which has to carry the weight of the quantum mechanical revolution. Probability theory involves highly structured statements about the nature of 'randomness', 'independence' and the like, concepts that are far from being trivially or immediately related to 'sense impressions'. It is much more likely that the enigmatic character of the E.P.R. experiment results from an ill-understood concept of 'statistical independence' than that is is an evidence for the collapse of a logical principle. (Of course, both these alternatives may be wrong and it may be that the principle of locality should carry the weight of the revolution.)

What I have been trying to accomplish in this thesis is a construction of a particular model for (a portion of) quantum statistics. It is a realistic model (or at least one can interpret it realistically) in that it assumes that particles together with their properties exist independently of our observations. It is a deterministic model in that it assumes that one can determine, in principle, the exact transformation of the particle properties which occurs during an interaction. Finally it is a local model in that it does not assume that particles communicate in an instantaneous, mysterious way.

The weight of the revolution is carried, in this model, by two related assumptions; firstly, that the distribution of traits is non-measurable and, secondly, that the calculation of probabilities is coordinate-dependent. The second assumption, in particular, is a radical one. In the E.P.R. experiment the probability for the event: "relative frequency converges to the quantum mechanical limit" equals one only when the calculation is carried out in a spherical coordinate system whose axis is one or the other of the directions in which the experiment *actually* takes place. I have not specified the physical mechanism or the reason why this occurs. It may have to do with the inexact nature of the apparatus (see section III.5) and it may have to do with another physical principle (such as a statistical violation of the isotropy of space). Whatever the explanation is, the model is consistent. Regardless of its truthfulness or physical plausibility I believe that the very possibility of such models strongly indicates that the problem of the interpretation of quantum mechanics and the relation between this theory and metaphysics is as unsettled an issue today as it has always been.

NOTES AND REFERENCES

Chapter I

1. Feynman (1965), p. 129.

2. Ibid., p. 123.

3. Heisenberg (1925). An English translation appeared in Van Der Waerden (1967).

4. Report of this incident in Kevles (1979), see also Forman (1971).

5. Personal recollection. Dirac made this comment during the Einstein Centennial Symposium held at the Van Lear Institute, Jerusalem, Israel, 1979.

6. Von Neuman (1955).

7. Bell, J. S. (1966), Böhm and Bub (1966), Bub (1974).

8. Forman (1971).

9. Kevles (1979).

10. Lacatos (1970).

11. Kline (1972).

12. Born (1948).

13. Feynman (1951, 1965), Feynman and Hibbs (1965).

14. Material on the theory of probability is taken by and large from Chow and Teicher (1978).

15. The point is that the means of a sequence of independent, identically distributed random variables converge to the expectation 'almost everywhere', not pointwise. Hence the term 'probability' appears in the formulation of the law of large numbers.

Chapter II

1. Feynman (1965), pp. 127-148.

2. For a classical wave theoretical analysis of diffraction and interference see Landau and Litshitz, 1975.

3. Feynman and Hibbs (1965).

4. Bell, J. S. (1964). Substential generalization of Bell's theorem are in Clauser and Horne (1974) Garg and Mermin (1982a,b).

5. Einstein, Podolsky and Rosen (1935).

6. Böhm (1951), pp. 614-619.

7. Stern and Gerlach (1922). For historical background see Jammer (1966).

8. Gibson and Pollard (1976), p. 218.

9. Streater and Wightman (1964). For more modern text see Lee (1981).

10. Electron polarization experiments are quite involved technically. I shall ignore experimental difficulties and present the results in schematic way.

11. The phenomena persists across very large distances. For the most recent results see Aspet, Delibard and Gérard (1982).

12. Mermin (1981).

13. Van Fraassen (1980).

14. Stapp (1975, 1977, 1980), d'Espangant (1976).

15. Capra (1975), Böhm (1981).

16. Quoted in Clark (1975).

17. Birkhoff and Von Neuman (1936).

18. Putnam (1968, 1976), Friedman and Putnam (1978).

19. Detailed suggestins in that spirit were made by Gudder (1969, 1973, 1980), Gudder and Zerbe (1981) whose approach is related to quantum logic. For a somewhat different view see Accardi (1981, 1982).

20. Portions of the model proposed in the following chapters appeared in Pitowsky (1982a, 1983a, 1983b). For comments and discussion see Pitowsky (1982c).

Chapter III

1. See e.g. Halmos (1950).

2. Rudin (1977) in particular theorem 15, p. 498.

3. In the construction I use the following fact: Let p be a real number $0 \le p \le 1$ then there is a sequence $A_1, A_2 \ldots A_n \ldots$ of measurable subsets of $[0,1]$ such that $m(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = p^k$ for all $k \ge 2$ and sequences of indices $1 \le i_1 < i_2 < \ldots < i_k < \infty$. (Here m is the Lebesgue measure on $[0,1]$).

4. Chow and Teicher (1978).

Chapter IV

1. Gibson and Pollard (1976).

Chapter V

1. Kochen and Specker (1976), Pitowsky (1982b).

Chapter VI

1. Klein (1972).

2. The following details regarding the axiom of choice are taken from Jech (1973).

3. Gödel (1940), Cohen (1966).

4. Solovey (1970).

5. This is due to Gödel's famous incompleteness theorem, Gödel (1931).

6. Cohen (1966).

7. Gödel (1940), as a Platonist, Gödel must commit himself to the existence (in-principle) of truth values, even for such abstract statements.

8. "I am coming more and more to the conviction that the necessity of our geometry cannot be demonstrated, at least neither by nor for, the human intellect . . . geometry should be ranked, not with arithmetic which is purely aprioristic, but with mechanics." Gauss (1817), quoted in Misner, Thorne and Wheeler (1970). See also Bell, E. T. (1973).

9. For details and critical discussion of both views see Fraenkel Bar-Hillel and Levy (1977), Dummett (1977).

10. See e.g., Ullman (1979) for further references.

11. I am using the term 'psychologically a priori' in the same sense as in Popper (1963).

12. Jech (1973).

13. Historical details are in Kline (1972).

14. For a survey of Wiener's contribution in that area see Kac (1966).

15. Putnam (1981). For a critical analysis and historical description of the model theoretic argument see Demopoulos and Friedman (1982).

16. A Shimony, private communication, N.D. Mermin, private communication.

17. For a survey of suggested alternatives to General Relativity see Misner, Thorne, Wheeler (1970).

18. Von Mieses (1957).

19. Keynes (1921).

20. Carnap (1950).

21. This task was carried out mainly by Schlick. See Schlick (1918).

22. Reference 19, Chapter II.

23. Quine (1953).

24. I suspect that the causes for this development are not entirely intrinsic to science. In particular, those who support the holistic non local interpretation openly admit that they were influenced by the 'cultural revolution' and the politics of protest in the sixties.

25. Van Fraassen (1980).

26. For details on these and related discoveries see Judson (1979).

27. Wittgenstein (1969).

28. See e.g., Accardi (1981) where he proves that the transition probabilities for any *pair* of observables can be recovered in a classical probablistic model.

29. Reichenbach (1949).

30. Birkhoff and Von Neuman (1936).

31. Wittgenstein (1969).

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